Journal of Computational and Applied Mechanics, Vol. 1., No. 1., (2000), pp. 13–21

STRESS FUNCTIONS FOR TORSION-FREE AXISYMMETRIC STATE OF STRESS

István Ecsedi
Department of Mechanics, University of Miskolc
3515 Miskolc – Egyetemváros, Hungary
mechecs@gold.uni-miskolc.hu

[Received: August 15, 1999]

Dedicated to Professor István Páczelt on the occasion of his sixtieth birthday

Abstract. The solution of problems of the theory of elasticity in terms of stresses requires the solution of equations of equilibrium. In two and three-dimensional application, the components of stress are frequently expressed in terms of partial derivatives of stress function, the correct expressions being chosen so that the conditions of static equilibrium become a consequence of partial derivatives independent of the order of differentiation. In this manner the solution of an equilibrium equation can be obtained without any difficulty. In this paper a systematic process is devised to derive the stress functions for the problems of torsion-free axisymmetric state of stresses. The applied method is based on the theory of total differentials. The stress boundary conditions are also formulated in terms of stress functions. The relations between the strain compatibility conditions and the stress functions are discussed. Different forms of solutions of equilibrium equations in terms of stress functions are also analyzed.

Keywords: Stress functions, axisymmetric, strain compatibility, virtual work

1. Introduction

For torsion-free axisymmetric state of stress the equilibrium equations can be written as

$$\frac{\partial}{\partial r}(r\sigma_r) + \frac{\partial}{\partial z}(r\tau_{rz}) - \sigma_\varphi + rq_r = 0 \qquad (r,z) \in T\ , \tag{1.1}$$

$$\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial}{\partial z}(r\sigma_z) + rq_z = 0 \qquad (r,z) \in T\ . \tag{1.2}$$

Here, we have used cylindrical coordinates (r, φ, z) with z as the axis of symmetry. In equations (1.1) and (1.2) $\sigma_r, \sigma_\varphi, \sigma_z$ are the normal stresses, τ_{rz} is the shearing stress and the body forces are denoted by q_r and q_z . All these quantities depend on the polar coordinates r and z only. We remark that for torsion-free axisymmetric state of stress the shearing stresses $\tau_{r\varphi}$ and $\tau_{z\varphi}$ vanish in all points of the body. In the sequel we shall assume that T is a simply connected plane region in the meridian section of the body of rotation and T has not any point in common with the axis z. The boundary of the meridian section T is the closed curve ∂T. The unit tangent and the outward unit normal to ∂T are denoted by $\mathbf{t}$ and $\mathbf{n}$, respectively. Let

$$R = R(s) \, , \qquad Z = Z(s) \qquad 0 \le s \le L \tag{1.3}$$

be the parametric equation of the curve ∂T, where s is the arc length measured on ∂T, and L is the total length of curve ∂T. The unit vectors in the radial and longitudinal directions are denoted by $\mathbf{e}_r$ and $\mathbf{e}_z$. It can be shown with ease that

$$\mathbf{t} = \frac{dR}{ds}\mathbf{e}_r + \frac{dZ}{ds}\mathbf{e}_z \, , \qquad \mathbf{n} = \frac{dZ}{ds}\mathbf{e}_r - \frac{dR}{ds}\mathbf{e}_z \tag{1.4}$$

and

$$t_r = -n_z = \frac{dR}{ds} \, , \qquad t_z = n_r = \frac{dZ}{ds} \, . \tag{1.5}$$

Let ∂T_p ($\partial T_p \in \partial T$) be the arc of ∂T on which tractions are imposed. The corresponding boundary conditions take the form

$$\sigma_r n_r + \tau_{rz} n_z = p_r \, , \qquad \tau_{rz} n_r + \sigma_z n_z = p_z \qquad (r,z) \in \partial T_p \tag{1.6}$$

where

$$\mathbf{p} = p_r(r,z)\mathbf{e}_r + p_z(r,z)\mathbf{e}_z \tag{1.7}$$

is prescribed on ∂T_p .

It is the purpose of the present paper to find the general solution in terms of stress functions to equations (1.1) and (1.2) and to analyze how the tractions and stress functions are related to each other. Langhaar and Stippes [3] presented a complete representation of stresses for axisymmetric stress states. The same problem was analyzed by Filonenko-Borodich [2]. The complete representation given in this paper is different from the solution derived by the two authors mentioned.

In Section 2 the stress function solution of equilibrium equations (1.1), (1.2) is derived and is shown to be complete. The degree of arbitrariness of the stress functions for a given state of stresses is then discussed. In Section 3 the traction boundary conditions are formulated in terms of stress functions. In Section 4 the stress function solution is derived from the principle of virtual work. Section 5 is devoted to the problem how the different solutions to the equilibrium equations in the terms of stress functions are related to each other. Section 6 contains some conclusions.

2. Stress Functions

In this paper we will not formulate explicitly the smoothness and continuity properties which are required. They may be deduced from known theorems of calculus (see, for example Courant [1], Rudin [4]).

We shall assume that the body forces can be represented as

$$rq_r = -\frac{\partial Q_r}{\partial r} \, , \qquad rq_z = -\frac{\partial Q_z}{\partial z} \tag{2.1}$$

where $Q_r = Q_r(r,z)$ and $Q_z = Q_z(r,z)$ are potential functions for the body forces q_r and q_z, respectively. Without loss of generality we can consider the stress component σ_φ as the partial derivative of a function $B = B(r,z)$ with respect to r, that is

$$\sigma_\varphi = \frac{\partial B}{\partial r} \qquad (r,z) \in T \cup \partial T \, . \tag{2.2}$$

Upon substitution of (2.1) and (2.2) into the equilibrium equations (1.1), (1.2) we obtain

$$\frac{\partial}{\partial r}(r\sigma_r - B - Q_r) + \frac{\partial}{\partial z}(r\tau_{rz}) = 0 \qquad (r,z) \in T \, , \tag{2.3a}$$

$$\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial}{\partial z}(r\sigma_z - Q_z) = 0 \qquad (r,z) \in T \, . \tag{2.3b}$$

The stress function solution of equations (1.1), (1.2) is supplied by the following theorem.

Theorem 2.1. Let the stresses be represented by

$$r\sigma_r = \frac{\partial^2 A}{\partial z^2} + B + Q_r \, , \tag{2.4a}$$

$$\sigma_\varphi = \frac{\partial B}{\partial r} \, , \tag{2.4b}$$

$$r\sigma_z = \frac{\partial^2 A}{\partial r^2} + Q_z \, , \tag{2.4c}$$

$$r\tau_{rz} = -\frac{\partial^2 A}{\partial r \partial z} \, , \tag{2.4d}$$

where $A = A(r,z)$ and $B = B(r,z)$ are arbitrary functions. This stress representation identically satisfies the equilibrium equations (1.1), (1.2).

The proof of this theorem can be obtained by direct substitution. The next theorem, which is motivated by stress representation (2.4a,b,c,d), states that if the body forces are obtainable from (2.1) then every solution of the equilibrium equations (1.1), (1.2) can be given by equations (2.4a,b,c,d).

Theorem 2.2. Let the stresses satisfy (1.1), (1.2). Then there exist functions $A = A(r,z)$ and $B = B(r,z)$ such that the stresses can be represented by equations (2.4a,b,c,d), and the stress functions $A = A(r,z)$ and $B = B(r,z)$ are single-valued.

Proof. According to the theory of total differentials (Courant [1], Rudin [4]), (2.3a) implies the existence of a single-valued function in $\overline{T} = T \cup \partial T$ such that

$$r\tau_{rz} = \frac{\partial a}{\partial r} \, , \qquad r\sigma_r - B - Q_r = -\frac{\partial a}{\partial z} \, . \tag{2.5}$$

Similarly, by quation (2.3b) there exists a function $b = b(r,z)$ in $\overline{T}$ such that

$$r\sigma_z - Q_z = \frac{\partial b}{\partial r} \, , \qquad r\tau_{rz} = -\frac{\partial b}{\partial z} \, . \tag{2.6}$$

From the equality of the two different expressions for τ_{rz} – these follow from $(2.5)_1$ and $(2.6)_2$ – we obtain

$$\frac{\partial a}{\partial r} + \frac{\partial b}{\partial z} = 0 \, . \tag{2.7}$$

A repeated application of the theory of total differentials gives that there exists a single-valued function $A = A(r,z)$ such that

$$a = -\frac{\partial A}{\partial z} \, , \qquad b = \frac{\partial A}{\partial r} \quad . \tag{2.8}$$

A combination of equations (2.5), (2.6) and (2.8) leads to stress representation (2.4a, b,c,d) which gives all the stress components. This last step completes the proof of Theorem 2.2. The degree of arbitrariness of the stress functions for a given state of stress is formulated in the following theorem.

Theorem 2.3. Let a given set of stresses which meet the equations (1.1), (1.2) be represented by (2.4a,b,c,d) in terms of the stress functions $A = A(r,z), B = B(r,z)$ and in terms of different stress functions $A' = A'(r,z), B' = B'(r,z)$. Then

$$A(r,z) = A'(r,z) + \alpha(z) + \beta_0 + \beta_1 r \, , \tag{2.9}$$

$$B(r,z) = B'(r,z) - \frac{d^2\alpha}{dz^2} \, , \tag{2.10}$$

where β_0, β_1 are arbitrary constants and $\alpha = \alpha(z)$ is an arbitrary function of z.

Proof. Using stress representation (2.4a,b,c,d) one finds that

$$\frac{\partial^2 A}{\partial z^2} + B + Q_r = \frac{\partial^2 A'}{\partial z^2} + B' + Q_r \, , \tag{2.11a}$$

$$\frac{\partial B}{\partial r} = \frac{\partial B'}{\partial r} \, , \tag{2.11b}$$

$$\frac{\partial^2 A}{\partial r^2} = \frac{\partial^2 A'}{\partial r^2} \, , \tag{2.11c}$$

$$\frac{\partial^2 A}{\partial r \partial z} = \frac{\partial A'}{\partial r \partial z} \, . \tag{2.11d}$$

Equation (2.11d) yields

$$A - A' = \alpha(z) + \beta(r) \tag{2.12}$$

where $\alpha = \alpha(z)$ and $\beta = \beta(r)$ are arbitrary functions, α depends only on z, and β depends only on r. Equation (2.11c) gives

$$\beta = \beta_0 + \beta_1 r \, , \tag{2.13}$$

where β_0, β_1 are the arbitrary constants. We obtain from equation (2.11b)

$$B = B' + \gamma(z) \, . \tag{2.14}$$

Inserting equations (2.12) and (2.14) into equation (2.11a) we arrive at

$$\gamma(z) = -\frac{d^2\alpha}{dz^2} \ . \tag{2.15}$$

which completes the proof of Theorem 2.3.

It is important to emphasize that the theorems proven are independent of any constitutive and compatibility equations which the stresses should also satisfy since we have been dealing with equilibrium equations only.

3. Stress boundary condition

In order to formulate the stress boundary conditions we start from equations (1.6) and (2.4a,b,c,d). By a simple substitution we obtain

$$\begin{aligned} r\sigma_r n_r + r\tau_{rz} n_z &= \frac{\partial^2 A}{\partial z^2} n_r + B n_r + Q_r n_r - \frac{\partial^2 A}{\partial r \partial z} n_z = \\ &= \frac{d}{ds}\left(\frac{\partial A}{\partial z}\right) + B\frac{dZ}{ds} + Q_r \frac{dZ}{ds} \qquad (r,z) \in \partial T_p \ , \end{aligned} \tag{3.1a}$$

$$\begin{aligned} r\tau_{rz} n_r + r\sigma_z n_z &= -\frac{\partial^2 A}{\partial r \partial z} n_r + \frac{\partial^2 A}{\partial r^2} n_z + Q_z n_z = \\ &= -\frac{d}{ds}\left(\frac{\partial A}{\partial r}\right) - Q_z \frac{dR}{ds} \qquad (r,z) \in \partial T_p \ . \end{aligned} \tag{3.1b}$$

Here we have used equation (1.5). Combination of equations (1.6) and (3.1a,b) leads to the result

$$\frac{d}{ds}\left(\frac{\partial A}{\partial z}\right) = r p_r - B\frac{dZ}{ds} - Q_r \frac{dZ}{ds} \ , \tag{3.2a}$$

$$-\frac{d}{ds}\left(\frac{\partial A}{\partial r}\right) = r p_z + Q_z \frac{dR}{ds} \ . \tag{3.2b}$$

After integrating these equations on ∂T_p we have

$$\left(\frac{\partial A}{\partial z}\right)_P - \left(\frac{\partial A}{\partial z}\right)_{P_0} = \int\limits_{\widehat{P_0 P}} r p_r \mathrm{d}s - \int\limits_{\widehat{P_0 P}} (B + Q_r)\frac{dZ}{ds}\mathrm{d}s \ , \tag{3.3a}$$

$$\left(\frac{\partial A}{\partial r}\right)_{P_0} - \left(\frac{\partial A}{\partial r}\right)_P = \int\limits_{\widehat{P_0 P}} r p_z \mathrm{d}s + \int\limits_{\widehat{P_0 P}} Q_z \frac{dR}{ds}\mathrm{d}s \ . \tag{3.3b}$$

In formulae (3.3a,b) the integrals are taken over an arc P_0P of ∂T_p. The lower limit P_0 is fixed and the upper limit P is regarded as a parameter. According to Theorem 2.3 we can set the starting values for the partial derivatives of the stress function A to

$$\left(\frac{\partial A}{\partial r}\right)_{P_0} = \left(\frac{\partial A}{\partial z}\right)_{P_0} = 0 \ . \tag{3.4}$$

4. Derivation of stress functions from the principle of virtual work

In this section we derive the solution of the homogeneous equilibrium equation in terms of stress functions from the principle of virtual work. The line of thought is based on that of Washizu [7]. The general solution of the homogeneous equilibrium equations is given in terms of stress functions A and B. In the absence of body forces one can write that

$$q_r = q_z = 0 \ , \qquad Q_r = Q_z = 0 \ . \tag{4.1}$$

The strain compatibility equations for torsion-free axisymmetric deformation are as follows [6]:

$$C_a = \frac{\partial^2 \varepsilon_r}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial r^2} - \frac{\partial^2 \gamma_{rz}}{\partial r \partial z} = 0 \qquad (r, z) \in T \ , \tag{4.2a}$$

$$C_b = \frac{\partial}{\partial r}(r\varepsilon_\varphi) - \varepsilon_r = 0 \qquad (r, z) \in T \ . \tag{4.2b}$$

Here $\varepsilon_r, \varepsilon_\varphi, \varepsilon_z$ are direct strains and γ_{rz} is the shear strain. The strain components depend on the radial coordinate r, and the longitudinal coordinate z. Washizu [7] proposes to introduce the strain compatibility conditions instead of the displacement components u and w into the principle of virtual work by the use of Lagrange multipliers. Since the problem is axisymmetric for the volume and the surface elements, one can write

$$\mathrm{d}V = 2\pi r \mathrm{d}T \ , \qquad \mathrm{d}S = 2\pi r \mathrm{d}s \ .$$

The principle of virtual work has the form

$$\int\limits_T r(\sigma_r \delta\varepsilon_r + \sigma_\varphi \delta\varepsilon_\varphi + \sigma_z \delta\varepsilon_z + \tau_{rz}\delta\gamma_{rz})\mathrm{d}T - \int\limits_{\partial T_p} r(p_r \delta u + p_z \delta w)\mathrm{d}s = 0 \ . \tag{4.3}$$

Here the infinitesimal virtual displacements $\delta u, \delta w$ and the infinitesimal virtual strains $\delta\epsilon_r, \delta\epsilon_\varphi, \delta\epsilon_z, \delta\gamma_{rz}$ should satisfy the strain-displacement relationships and the homogeneous geometrical boundary conditions imposed on the boundary segment ∂T_u . Consequently

$$\delta u = \delta w = 0 \qquad \text{on} \ \ \partial T_u \ . \tag{4.4}$$

We remark that

$$\partial T_p \cup \partial T_u = \partial T \quad \text{and} \quad \partial T_p \cap \partial T_u = \{\emptyset\} \ .$$

We apply equations (4.2a,b) as the field conditions of compatibility instead of the strain-displacement relationship. We can now transform equation (4.3) into the form

$$\int\limits_T r(\sigma_r \delta\varepsilon_r + \sigma_\varphi \delta\varepsilon_\varphi + \sigma_z \delta\varepsilon_z + \tau_{rz}\delta\gamma_{rz})\mathrm{d}T - \int\limits_T (\lambda_a \delta C_a + \lambda_b \delta C_b)\mathrm{d}T + \{\text{boundary terms}\} = 0 \tag{4.5}$$

where λ_a and λ_b are the Lagrange multipliers which depend on the coordinates r and z. After some calculations including repeated partial integrations, equation (4.5) is manipulated into its final form:

$$\int_T \left[\left(r\sigma_r - \frac{\partial^2\lambda_a}{\partial z^2} - \lambda_b\right)\delta\varepsilon_r + r\left(\sigma_\varphi - \frac{\partial\lambda_b}{\partial r}\right)\delta\varepsilon_\varphi + \right.$$
$$\left. + \left(r\sigma_z - \frac{\partial^2\lambda_a}{\partial r^2}\right)\delta\varepsilon_z + \left(r\tau_{rz} + \frac{\partial^2\lambda_a}{\partial r\partial z}\right)\delta\gamma_{rz}\right]\mathrm{d}T + \{\text{boundary terms}\} = 0\,. \quad (4.6)$$

Since the variations $\delta\varepsilon_r, \delta\varepsilon_\varphi, \delta\varepsilon_z$ and $\delta\gamma_{rz}$ are arbitrary, we have

$$r\sigma_r = \frac{\partial^2\lambda_a}{\partial z^2} + \lambda_b\,, \qquad \sigma_\varphi = \frac{\partial\lambda_b}{\partial r}\,,$$
$$r\sigma_z = \frac{\partial^2\lambda_a}{\partial r^2}\,, \qquad r\tau_{rz} = -\frac{\partial^2\lambda_a}{\partial r\partial z}\,. \quad (4.7)$$

A comparison of equations (4.7) and (2.4a,b,c) – in the latter case $Q_r = Q_z = 0$ – shows that

$$\lambda_a = A \qquad \text{and} \qquad \lambda_b = B\,, \quad (4.8)$$

thus the Lagrange multipliers λ_a and λ_b are stress functions.

5. Comparison of various formulations

It follows from the general axisymmetric solution of the homogeneous equilibrium equations (due to symmetry $\tau_{r\varphi} = \tau_{\varphi z} = 0$) given by Filonenko-Borodich that

$$r\sigma_r = \frac{\partial^2 f_1}{\partial z^2}\,, \quad (5.1a)$$
$$\sigma_\varphi = \frac{\partial^2 f_2}{\partial z^2}\,, \quad (5.1b)$$

$$r\sigma_z = -\frac{\partial}{\partial r}\left(f_2 - \frac{\partial f_1}{\partial r}\right)\,, \quad (5.1c)$$
$$r\tau_{rz} = \frac{\partial}{\partial z}\left(f_2 - \frac{\partial f_1}{\partial r}\right)\,. \quad (5.1d)$$

Here, $f_1 = f_1(r,z)$ and $f_2 = f_2(r,z)$ are stress functions. Formulas (5.1a,b,c,d) can be obtained from formula (1.10) of paper [2] by putting $f_3 = 0$. The next theorem relates the stress functions $f_1 = f_1(r,z)$, $f_2 = f_2(r,z)$ to the stress functions $A = A(r,z), B = B(r,z)$ assuming that the stress state is the same.

Theorem 5.1. If the stress functions $f_1 = f_1(r,z)$, $f_2 = f_2(r,z)$ and $A = A(r,z)$, $B = B(r,z)$ produce the same stress state then the following equations hold

$$f_1 = A + b\ , \qquad f_2 = \frac{\partial b}{\partial r}\ , \qquad \frac{\partial^2 b}{\partial z^2} = B\ . \tag{5.2}$$

In the absence of body forces and assuming an axisymmetric stress state, H. L. Langhaar and M. Stippes also gave a solution in terms of stress functions for the equilibrium equations [3]:

$$\sigma_r = \frac{\partial^2 F}{\partial z^2} + \frac{1}{r}\frac{\partial H}{\partial r}\ , \tag{5.3a}$$

$$\sigma_\varphi = \frac{\partial^2 F}{\partial z^2} + \frac{\partial^2 H}{\partial r^2}\ , \tag{5.3b}$$

$$\sigma_z = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r}\ , \tag{5.3c}$$

$$\tau_{rz} = -\frac{\partial^2 F}{\partial r \partial z}\ . \tag{5.3d}$$

Theorem 5.2 gives the connection between the stress functions $F = F(r,z)$, $H = H(r,z)$ and $A = A(r,z), B = B(r,z)$.

Theorem 5.2. If the stress state is the same, then the following equations hold between the stress functions $F = F(r,z), H = H(r,z)$ and $A = A(r,z), B = B(r,z)$:

$$A(r,z) = r\frac{\partial f}{\partial r} - f\ , \tag{5.4a}$$

$$B(r,z) = \frac{\partial^2 f}{\partial z^2} + \frac{\partial H}{\partial r}\ , \tag{5.4b}$$

where

$$F = \frac{\partial f}{\partial r}\ . \tag{5.5}$$

The proof of Theorem 5.1 and 5.2 can be obtained from the comparison of the various stress function solutions (4.7), (4.8), (5.1a,b,c,d) and (5.3a,b,c,d).

It is obvious that the substitution of the expressions giving f_1 and f_2 in terms of A, b and B into the Filonenko-Borodich solution leads immediately to the solution of the homogeneous equilibrium equation established in this paper.

A similar statement can be formulated for the Langhaar-Stippes solution. In the absence of body forces the stress representation suggested in this paper leads to the Langhaar-Stippes solution in terms of stress functions $F = F(r,z)$ and $H = H(r,z)$ if we use the stress functions $A = A(r,z)$ and $B = B(r,z)$ given by (5.4a,b), (5.5).

6. Conclusions

In this paper the general solution of equilibrium equation is presented for torsion-free axisymmetric state of stress. It has been shown that the solution given is complete and its degree of arbitrariness is also analyzed. The stress boundary conditions in terms of stress functions are also given. In the absence of body forces the stress representation we have found – solution to the homogenous equilibrium equations in terms of stress functions – can also be derived from the principle of virtual work. The presented stress function solution is then compared with other stress function solutions. The solution of equilibrium equations for axisymmetric torsion-free state of stress is very similar to the Airy solution of equilibrium equations for plane problems [5]. The results presented in the paper are all independent of any constitutive and compatibility equations which the stresses should satisfy in order to be the solution of a given boundary-value problem. If the body is elastic, then the field equations the stress functions should meet are the Beltrami-Mitchell equations under the prescribed boundary conditions. The stress functions solution of equilibrium equations gives a possibility to use the variational method elaborated by A. Castigliano. Another field of applications is the formulation of stress-based finite element models.

Acknowledgement. This work was supported by the Hungarian Ministry of Education under grant FKFP OO40/1999.

REFERENCES

1. Courant, R.: *Differential and Integral Calculus*, Vol. II, Interscience, New York, 1939.
2. Filonenko-Borodich, M. M.: *Some generalisation of Lamé's problem for an elastic parallepiped*, Prikl. Matem. i Mekh., **17** (4), (1953), 465-469.
3. Langhaar, H. L. and Stippes, M.: *Three-dimensional stress functions*, Journal Franklin Inst., **258** (5), (1954), 371-382.
4. Rudin, W.: *Principles of Mathematical Analysis*, 2nd Edition., McGraw–Hill. New York. 1961.
5. Timoshenko, S. and Goodier, J. N.: *Theory of Elasticity*, McGraw-Hill. New-York, 1952.
6. Vasiljev, B. and Chernih, B. A. and Koltunov, M. A.: *Elasticity and Strength of Cylindrical Bodies*, Higher School, Moscow, 1975. (in Russian)
7. Washizu, K.: *Variational Methods in Elasticity and Plasticity*, Second Ed., Pergamon Press, New York, 1975.