THREE DIMENSIONAL STOCHASTIC MODEL OF TURBULENCE

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Abstract. Mean velocity distributions of fully developed turbulent flows show a similarity revealed by experiments. The theoretical approach of *Kármán's* similarity hypothesis [1] concerning two-dimensional turbulent velocity fluctuations was based on experiments, but it was criticized on the basis of the generally accepted view that the turbulence phenomena was always three-dimensional. This paper intends to reveal that the mechanical similarity hypothesis is applicable to the 3-D boundary layer flow and a 3-D model of turbulence will be shown according to which the inner mechanism of the turbulence can be represented by a stochastic process including 5 independent probability variables.

Keywords: Similarity hypothesis, stochastic, turbulence model, turbulent vortex diffusion, Reynolds' stress tensor, vector potential of turbulent velocity fluctuation.

1. Governing equations of turbulent motion

In the *Eulerian* description of the turbulent motion of a fluid, continuum can be composed by the superposition of two velocity fields depending on the space coordinates and time. The first is the very rapid stochastic velocity fluctuation in time, which is the turbulence phenomenon. The second one is the time dependent mean velocity describing the motion of the continuum. All the characteristics of the motion can be composed of two components: *instantaneous value = mean value + fluctuation.* The mean value of a variable in a time interval t_0 is

$$\Phi(\mathbf{r},t) = \frac{1}{t_0}\int_{t}^{t+t_0} \Phi_T(\mathbf{r},\tau)\,\mathrm{d}\tau \tag{1.1}$$

where Φ can be scalar, vector or tensor field, and the subscript T refers to the turbulent instantaneous value. Accordingly, the instantaneous value of the velocity field has the form:

$$\mathbf{v}_T(\mathbf{r},t) = \mathbf{v}(\mathbf{r},t) + \mathbf{v}'(\mathbf{r},t)\,,$$

where $\mathbf{v}(\mathbf{r},t)$ is the turbulent mean value of the motion taken on a sufficiently long interval of t_0 compared to the periodicity of the fluctuation, and $\mathbf{v}'(\mathbf{r},t)$ is the turbulent velocity fluctuation, the mean value of which is zero. Due to the velocity fluctuation in the turbulent flow, a very intensive change of momentum takes place, increasing the resistance against the deformation of the flowing fluid. In other words, the apparent viscosity of the flowing fluid will be increased due to the change of momentum.

This study will be restricted to isothermal motion of incompressible fluids. The equation of continuity concerning the instantaneous velocity field $\mathbf{v}_T(\mathbf{r},t)$ is as follows:

$$\nabla \cdot \mathbf{v}_T = 0\,. \tag{1.2}$$

The *Stokes* molecular viscosity law is valid for the instantaneous turbulent motion as is commonly accepted. Consequently, the turbulent stress tensor can be written for the instantaneous values:

$$\mathbf{F}_T = -p_T\mathbf{I} + \boldsymbol{\sigma}_T = -p_T\mathbf{I} + \eta\,\left(\mathbf{v}_T \circ \nabla + \nabla \circ \mathbf{v}_T\right)\,, \tag{1.3}$$

where p_T is the instantaneous pressure, $\boldsymbol{\sigma}_T$ is the deviator of the instantaneous stress tensor $\mathbf{F}_T$, η is the dynamic viscosity and $\mathbf{I}$ is the unit tensor. If the force field is derivable from a potential U, the *Navier-Stokes* momentum equation for the viscous turbulent flow with instantaneous quantities takes the form

$$\frac{\partial \mathbf{v}_T}{\partial t} + (\mathbf{v}_T \cdot \nabla)\,\mathbf{v}_T = -\nabla\,U + \frac{1}{\rho}\,\mathrm{Div}\left[-p_T\mathbf{I} + \boldsymbol{\sigma}_T\right]. \tag{1.4}$$

Here and in the sequal Div [] = []$\cdot\nabla$. Introducing the vortex vector $\boldsymbol{\Omega}_T = \nabla\times\,\mathbf{v}_T$ and taking the curl of the previous equation, we obtain the *Helmholz-Thomson* vortex theorem for the instantaneous velocity field:

$$\frac{\partial \boldsymbol{\Omega}_T}{\partial t} + (\mathbf{v}_T \cdot \nabla)\,\boldsymbol{\Omega}_T - (\boldsymbol{\Omega}_T \cdot \nabla)\,\mathbf{v}_T = \nu \triangle \boldsymbol{\Omega}_T\,. \tag{1.5}$$

Let $\mathbf{a}(\mathbf{r},t)$ be a given vector field in the velocity field $\mathbf{v}(\mathbf{r},t)$. The necessary and sufficient condition for the vector lines to satisfy equation $\mathbf{a}\times\,\mathrm{d}\mathbf{r} = \mathbf{0}$ and to be constituted by the same fluid particles during the motion, and for the intensity of the vector tubes $\mathbf{a}\cdot\mathrm{d}\mathbf{A} = a\mathrm{d}A_n$ to remain constant is:

$$\frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{a} - (\mathbf{a} \cdot \nabla)\,\mathbf{v} + \mathbf{a}\,(\nabla \cdot \mathbf{v}) = \mathbf{0}\,.$$

This equation is called *Friedman* conservation law of vector tubes [2]. Consequently, according to equation (1.5) the vortex lines $(\nabla \times \mathbf{v}_T)\times\mathrm{d}\mathbf{r} = \mathbf{0}$ in an incompressible fluid flow with potential force field are not conserved but vanish, i.e., diffuse in the surroundings. The measure of diffusion is determined by the term on the right-hand side (RHS) of equation (1.5), i.e., by $\nu\,\Delta\,\boldsymbol{\Omega}_T$. In case of inviscid fluid for which $\nu \to 0$ the vortex lines are conserved.

The equation of continuity for an incompressible fluid can be given in terms of the velocity field $\mathbf{v}(\mathbf{r},t)$:

$$\nabla \cdot \mathbf{v} = \mathbf{0}. \tag{1.6}$$

The time-mean value of equation (1.4) is to be taken to obtain the *Reynolds* momentum equation for the turbulent motion of fluid particles

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\,\nabla\,U + \frac{1}{\rho}\,\mathrm{Div}\,\left[-p\mathbf{I} + \boldsymbol{\sigma} - \rho\left(\overline{\mathbf{v}' \circ \mathbf{v}'}\right)\right], \tag{1.7}$$

where the time-mean value is denoted by an overbar, $\boldsymbol{\sigma}$ is the time-mean value of the stress tensor given by the *Stokes* relation

$$\boldsymbol{\sigma} = \eta \ (\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v}) \ . \tag{1.8}$$

The last expression in the bracket on the RHS of equation (1.7) is the apparent turbulent stress tensor named after *Reynolds*: $\mathbf{F}_R = -\rho \left(\overline{\mathbf{v}' \circ \mathbf{v}'} \right)$.

The vortex theorem for the vector $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$ in the mean velocity field $\mathbf{v}(\mathbf{r},t)$ determining the phenomena of the vortex lines $(\nabla \times \mathbf{v}) \times \mathrm{d}\mathbf{r} = \mathbf{0}$ follows from equation (1.5) by taking its time-mean value:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + (\mathbf{v} \cdot \nabla) \boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \nabla) \mathbf{v} = \nu \triangle \boldsymbol{\Omega} + \nabla \times \left(\overline{\mathbf{v}' \times \boldsymbol{\Omega}'} \right) . \tag{1.9}$$

Here $\boldsymbol{\Omega}' = \nabla \times \mathbf{v}'$ is the vortex field of the velocity fluctuation. Consequently, vortex lines in the turbulent mean velocity field $\mathbf{v}(\mathbf{r},t)$ are not conserved even in the extreme case when $\nu \to 0$, but diffuse. The measure of vortex diffusion is determined by the viscosity of fluid and dominantly by the velocity fluctuation of turbulence.

2. Vector Potential of Turbulent Velocity Fluctuation

Let P be a fixed arbitrary point, $\mathbf{v}_P$ the turbulent mean velocity and $\boldsymbol{\Omega}_P = \nabla \times \mathbf{v}_P$ the vortex vector in the flow. Furthermore let Q be a *varying* point in the vicinity of P in which the turbulent instantaneous values are

$$\mathbf{v}_T = \mathbf{v}_Q + \mathbf{v}' \qquad \text{and,} \qquad \boldsymbol{\Omega}_T = \nabla \times \mathbf{v}_T = \boldsymbol{\Omega}_Q + \boldsymbol{\Omega}'$$

where $\mathbf{v}_Q$ and $\boldsymbol{\Omega}_Q$ are time-mean values, $\mathbf{v}'$ and $\boldsymbol{\Omega}' = \nabla \times \mathbf{v}'$ are the turbulent fluctuations. Because the order of the change in time of the fluctuations $\mathbf{v}'$ is much greater than that of the mean velocity, the error will not be significant supposing the mean value field together with $\mathbf{v}_P$ and $\boldsymbol{\Omega}_P$ are *constants*. The *Helmholz-Thomson* vortex theorem (1.5) is obviously valid in a relative coordinate system with its origin in P and moving with the constant turbulent mean velocity $\mathbf{v}_P$. Consequently, it can be applied to the vortex field $\boldsymbol{\Omega}_T$ in the velocity field moving with $\mathbf{v}_T - \mathbf{v}_P$, and we get:

$$\frac{\partial \boldsymbol{\Omega}_T}{\partial t} + [(\mathbf{v}_T - \mathbf{v}_P) \cdot \nabla] \boldsymbol{\Omega}_T - (\boldsymbol{\Omega}_T \cdot \nabla)(\mathbf{v}_T - \mathbf{v}_P) = \nu \triangle \boldsymbol{\Omega}_T .$$

To examine the turbulent motion let us substitute the instantaneous values of $\mathbf{v}_T$ and $\boldsymbol{\Omega}_T$ and take the limit $Q \to P$, $\mathbf{v}_T \to \mathbf{v}_P$ and $\boldsymbol{\Omega}_T \to \boldsymbol{\Omega}_P$ to obtain the differential equation:

$$\frac{\partial \boldsymbol{\Omega}'}{\partial t} + \left(\mathbf{v}' \cdot \nabla \right) \boldsymbol{\Omega}' - \left(\boldsymbol{\Omega}' \cdot \nabla \right) \mathbf{v}' = \nu \triangle \boldsymbol{\Omega}' + (\boldsymbol{\Omega} \cdot \nabla) \mathbf{v}' . \tag{2.1}$$

As P is arbitrary, the subscript at $\boldsymbol{\Omega}$ can be omitted without causing any misunderstanding. Equation (2.1) is a *vortex theorem for the turbulent velocity fluctuation field* $\mathbf{v}'$ meaning that the vector lines determined by the differential equation $\left(\nabla \times \mathbf{v}' \right) \times \mathrm{d}\mathbf{r} = \mathbf{0}$ are not conserved even if $\nu \to 0$ but scattered and diffused while moving.

The effect of molecular viscosity will decrease due to the change of momentum amongst the fluid particles in the turbulent flow; therefore, the term $\nu\triangle\mathbf{\Omega}^{'}$ on the RHS of equation (2.1) becomes negligible. The boundary layer flows where the shear stress is dominant and $\mathbf{v}$ and $\nabla\times\ \mathbf{v}$ are not parallel will be examined in a suitable coordinate system. An orthogonal curvilinear coordinate system $q_1^{'}$, $q_2^{'}$, $q_3^{'}$ will be attached to the mean velocity field as well, for which the base vectors are determined by $\mathbf{v}$ and $\nabla\times\ \mathbf{v}$ – see Figure 1 – as follows:

$$\mathbf{e}_3^{'}=\frac{\nabla\times\ \mathbf{v}}{|\nabla\times\ \mathbf{v}|};\ \mathbf{e}_2^{'}=\frac{\mathbf{v}\times\ \nabla\times\ \mathbf{v}}{|\mathbf{v}\times\ \nabla\times\ \mathbf{v}|};\ \mathbf{e}_1^{'}=\mathbf{e}_2^{'}\times\mathbf{e}_3^{'}=\frac{1}{\sqrt{1-\lambda^2}}\left(\frac{\mathbf{v}}{|\mathbf{v}|}-\lambda\frac{\nabla\times\ \mathbf{v}}{|\nabla\times\ \mathbf{v}|}\right),$$

where

$$\lambda=\frac{\mathbf{v}}{|\mathbf{v}|}\cdot\frac{\nabla\times\mathbf{v}}{|\nabla\times\mathbf{v}|}.$$

This is called the natural coordinate system of the boundary layer flow. The turbulent mean velocity can be expressed by its components in the $q_1^{'}$, $q_2^{'}$, $q_3^{'}$ system:

$$\mathbf{v}=v_{1'}\cdot\mathbf{e}_1^{'}+v_{3'}\cdot\mathbf{e}_3^{'}=v\sqrt{1-\lambda^2}\mathbf{e}_1^{'}-\lambda\, v\,\mathbf{e}_3^{'}$$

Here $v=\sqrt{v_{1'}^2+v_{3'}^2}$ is the absolute value of the velocity. The vorticity of the mean velocity can be written as

$$\mathbf{\Omega}=\nabla\times\mathbf{v}=-\left|\nabla\times\mathbf{v}\right|\mathbf{e}_3^{'}=-\mathit{\Omega}\mathbf{e}_3^{'}, \tag{2.2}$$

where $\mathit{\Omega}$ is the rate of change of the $q_1^{'}$ component of the velocity in the direction $q_2^{'}$. One can see in Figure 1 that $\mathit{\Omega}=-\mathit{\Omega}_{3'}$ is the negative component of $\mathbf{\Omega}$ in the direction $q_3^{'}$:

$$\mathit{\Omega}=-\mathit{\Omega}_{3'}=\frac{1}{H_1^{'}H_2^{'}}\frac{\partial\left(v_{1'}H_1^{'}\right)}{\partial q_2^{'}}, \tag{2.3}$$

where $H_i^{'}$ $(i=1,2,3)$ are the Lamé's metric coefficients in the natural coordinate system $q_1^{'}$, $q_2^{'}$, $q_3^{'}$. One can easily see that if $\mathbf{v}$ and $\nabla\times\ \mathbf{v}$ are perpendicular vectors as is true for two-dimensional flows, then $\mathbf{v}\cdot\mathbf{\Omega}=0$, and therefore $\mathbf{e}_1^{'}=\frac{\mathbf{v}}{|v|}$, i.e., the streamlines coincide with the coordinate $q_1^{'}$.

Making use of equation (2.1) the turbulent fluctuation can be investigated separately from the mean flow in a coordinate system moving with the mean velocity $\mathbf{v}$. The base vectors in thes coordinates sysytem are the same as those we introduced earlier. In this system the differential operator ∇ and the product $\nabla\cdot\mathbf{\Omega}$ can be expressed as follows:

$$\nabla=\frac{\mathbf{e}_1^{'}}{H_1^{'}}\frac{\partial}{\partial q_1^{'}}+\frac{\mathbf{e}_2^{'}}{H_2^{'}}\frac{\partial}{\partial q_2^{'}}+\frac{\mathbf{e}_3^{'}}{H_3^{'}}\frac{\partial}{\partial q_3^{'}}\qquad\text{and}\qquad\mathbf{\Omega}\cdot\boldsymbol{\nabla}=-\mathit{\Omega}\left(\mathbf{e}_3^{'}\cdot\nabla\right)=-\frac{\mathit{\Omega}}{H_3^{'}}\frac{\partial}{\partial q_3^{'}}\ .$$

By applying the vortex theorem (2.1) in the $q_1^{'}$, $q_2^{'}$, $q_3^{'}$ coordinate system moving with the flow for the velocity fluctuation $\mathbf{v}^{'}$ we obtain the following differential equation

$$\frac{\partial\left(\nabla\times\ \mathbf{v}^{'}\right)}{\partial t}+\left(\mathbf{v}^{'}\cdot\nabla\right)\left(\nabla\times\ \mathbf{v}^{'}\right)-\left(\left(\nabla\times\mathbf{v}^{'}\right)\cdot\nabla\right)\mathbf{v}^{'}=-\frac{\mathit{\Omega}}{H_3^{'}}\frac{\partial\mathbf{v}^{'}}{\partial q_3^{'}}. \tag{2.4}$$

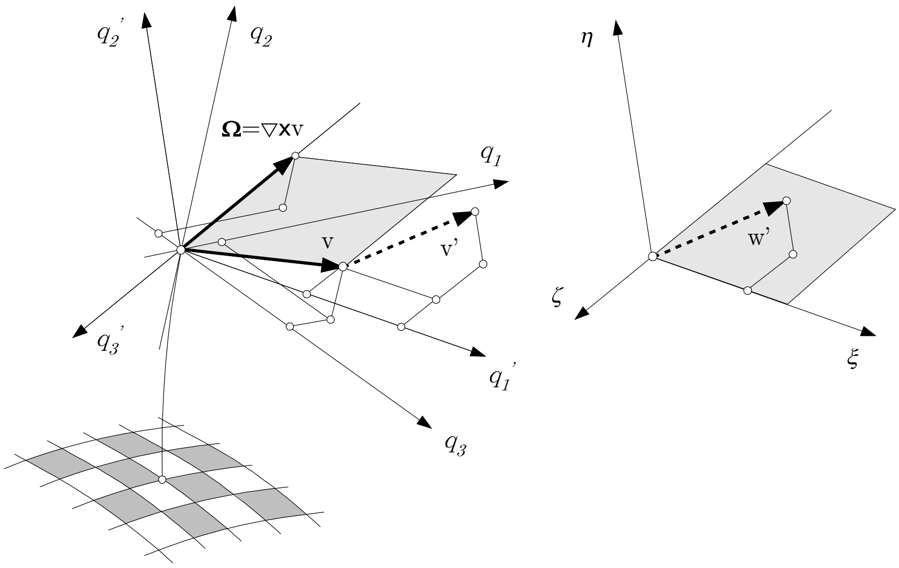

Figure 1. The boundary layer flow in the natural orthogonal curvilinear coordinate system

This equation is considered as the *momentum equation of turbulence* describing the fluctuation in the moving coordinate system. The sole dependence is obvious from equation (2.4) relating the turbulent fluctuation $\mathbf{v}^{'}$ to the vorticity of the turbulent mean velocity $\mathbf{v}$. Consequently the dependence between $\mathbf{F}_R = -\rho\left(\overline{\mathbf{v}' \circ \mathbf{v}'}\right)$, the apparent *Reynolds*' stress tensor and the vorticity of the mean velocity is obvious and *no relation exists* between the stress and deformation velocity tensors [1] .

As the turbulent fluctuation velocity $\mathbf{v}^{'}(\mathbf{r}, t)$ is a rotational vector field there exists a vector potential function $\mathbf{\Psi}(\mathbf{r}, t)$ from which v' can be obtained as $\mathbf{v}' = \nabla \times \mathbf{\Psi}$. Supposing that the vector potential is sourceless, as is commonly supposed, we obtain $\nabla \times v' = \nabla \times (\nabla \times \mathbf{\Psi}) = -\triangle \mathbf{\Psi}$ and we arrive at the following differential equation for the vector potential $\mathbf{\Psi}$ by substituting it into equation (2.4):

$$\frac{\partial \triangle \mathbf{\Psi}}{\partial t} + ((\nabla \times \mathbf{\Psi}) \cdot \nabla) \triangle \mathbf{\Psi} - (\triangle \mathbf{\Psi} \cdot \nabla)(\nabla \times \mathbf{\Psi}) = \Omega \left(\mathbf{e}_3^{'} \cdot \nabla\right)(\nabla \times \mathbf{\Psi}) \ . \qquad (2.5)$$

The character of this equation is that of a momentum equation such as (2.4) so it is suitable to investigate the similarity phenomena of turbulent motion. Two patterns of motion are similar to each other if the momentum equations of each can be transformed by geometrical and dynamic transformations. For the moving $q_1^{'}$, $q_2^{'}$, $q_3^{'}$ orthogonal curvilinear coordinate system, we introduce the following geometrical transformation

$$H_1^{'} dq_1^{'} = \ell \ d\xi; \ \ H_2^{'} dq_2^{'} = \ell \ d\eta; \ H_3^{'} dq_3^{'} = \ell \ d\zeta \ ,$$

[1] The equation of continuity (1.6) and the Reynolds' momentum equation (1.7) together with Equ. (2.4) is a closed differential equation system of seven equations and there are seven unknown functions, namely the three mean velocity components, the three velocity fluctuation components and the pressure to be determined.

which transforms the physical space of the velocity fluctuation $\mathbf{v}^{'}(\mathbf{r},t)$ to the points of an orthogonal coordinate system where the fixed point P corresponds to the origin O(0,0,0) (Figure 1) in the coordinates ξ,η,ζ. Let us introduce the transformations $dt = T d\tau$ for the time and $\mathbf{\Psi}\left(q_1^{'},q_2^{'},q_3^{'},t\right) = C\cdot\,\mathbf{f}\,(\xi,\eta,\zeta,\tau)$ for the vector potential. Here ℓ is the length scale, T is the time scale and C is the scale of vector potential. The last three scales are independent of the $q_1^{'},q_2^{'},q_3^{'}$ and ξ,η,ζ coordinates. These transformations will be substituted to the (2.5) momentum equation. By this transformation the turbulent motion is transformed into the ξ,η,ζ orthogonal coordinate system, therefore it is called the *map of turbulence*.

The turbulent stream patterns in the points of the flow can be considered similar if the differential equation for the dimensionless vector potential $\mathbf{f}$ transformed from the differential equation (2.5) for $\mathbf{\Psi}$ does not depend on the specification of motion in point P. It is easy to see that the similarity conditions are as follows:

$$\frac{C}{\ell^2 T} = \frac{C^2}{\ell^4} = \frac{C\,\Omega}{\ell^2}$$

If these conditions are fulfilled, then the transformed equation of motion (2.5) for the vector potential $\mathbf{f}$ takes the form

$$\frac{\partial \Delta \mathbf{f}}{\partial \tau} + \left(\left(\nabla \times \mathbf{f}\right)\cdot\nabla\right)\Delta\mathbf{f} - \left(\Delta\mathbf{f}\cdot\nabla\right)\left(\nabla \times \mathbf{f}\right) = \frac{\partial}{\partial \zeta}\left(\nabla \times \mathbf{f}\right) \tag{2.6}$$

The last equation does obviously not depend on the characteristics of flow in a point P. Consequently, the turbulent velocity fluctuation $\mathbf{v}^{'}(\mathbf{r},t)$ in the points of the boundary layer flow mapped to the space of ξ,η,ζ will have the same $\mathbf{f}\,(\xi,\eta,\zeta,\tau)$ vector potential, which is exactly the expression of the mechanical similarity for the three dimensional turbulence phenomena. From these three conditions of similarity for ℓ, T and C, only one can be selected freely because of the relationship amongst them. Selecting one of them, the two remaining are determined. Let us select the length scale ℓ we get for the left two scale factors:

$$T = \frac{1}{\Omega}\ ;\ C = \ell^2\Omega.$$

As a consequence of the mechanical similarity hypothesis in the point ξ,η,ζ of the mapped space of three dimensional turbulence, the vector potential $\mathbf{f}\,(\xi,\eta,\zeta,\tau)$ determines the structure of turbulence up to a certain measure of dimensionless size, and converting it by the length scale ℓ, the result will be the true for the local turbulent motion corresponding to the geometrical circumstances. The length scale ℓ characterizes the size of the local turbulence.

3. Three Dimensional Stochastic Model of Turbulence

It is clear from previous considerations that the partial differential equation (2.6) for $\mathbf{f}\,(\xi,\eta,\zeta,\tau)$ is equivalent to the creation of the inner mechanism of turbulence models. The particular solutions of equation (2.6) lead to different models for turbulence.

Since direct physical tests theoretically cannot be carried out, only the numerical results computed on the base of the models can justfy them. It is always advisable to make simplifications when seeking particular solutions. Therefeore we shall make the following restrictions when seeking the particular solutions of equation (2.6):

$$\nabla \cdot \mathbf{f} = 0 \qquad \text{and} \qquad \Delta\, \mathbf{f} = a\, \mathbf{f} + b\, \nabla \times \mathbf{f}\,.$$

Here a and b are scalars. Since the effect of molecular viscosity can be excluded in the mapped space of turbulence, we suppose that the vector lines satisfying the equation $\mathbf{f} \times d\mathbf{r} = \mathbf{0}$ will remain in this space. By introducing the notation $\mathbf{w}' = \nabla \times \mathbf{f}$ *Friedman*'s law of conservation of vector tubes becomes:

$$\frac{\partial \mathbf{f}}{\partial \tau} + (\mathbf{w}' \cdot \nabla)\, \mathbf{f} - (\mathbf{f} \cdot \nabla)\, \mathbf{w}' = \mathbf{0}\,. \tag{3.1}$$

The equation to be solved under these conditions is as follows:

$$\frac{\partial}{\partial \tau} (\nabla \times \mathbf{f}) = \omega\, \frac{\partial}{\partial \zeta} (\nabla \times \mathbf{f})\ . \tag{3.2}$$

Here $\omega = 1/b$. One can see that only $\nabla \times \mathbf{f}$ is sought and not the vector potential $\mathbf{f}$ itself and by this the inner mechanism of turbulence can be determined.

A possible particular solution, which characterizes the mechanism by a stochastic process, for example may be following:

$$\nabla \times \mathbf{f}\ \ (\xi, \eta, \zeta, \tau) = \varepsilon \sum_{n=1}^{N} \begin{pmatrix} C_{1n} e^{n\xi} \cos\left[n\left(\zeta + \omega\tau\right) + \alpha_{1n}\right] \\ C_{2n} \cos\left[n\left(\zeta + \omega\tau\right) + \alpha_{2n}\right] \\ C_{3n} e^{-n\eta} \cos\left[n\left(\zeta + \omega\tau\right) + \alpha_{3n}\right] \end{pmatrix} - \\ - \varepsilon \sum_{n=1}^{N} \begin{pmatrix} C_{2n} \sin\left[n\left(\zeta + \omega\tau\right) + \alpha_{2n}\right] \\ C_{3n} e^{-n\eta} \sin\left[n\left(\zeta + \omega\tau\right) + \alpha_{3n}\right] \\ C_{1n} e^{n\xi} \sin\left[n\left(\zeta + \omega\tau\right) + \alpha_{1n}\right] \end{pmatrix}. \tag{3.3}$$

The parameters ε and ω are optional. The meaning of ω is the lowest angular frequency of the turbulent fluctuation. The C_{in} $(i = 1, 2, 3)$ are the random amplitudes of the wave components, which are probability variables with uniform probability distributions in the given $[0, \delta_i]$ intervals. The probability variables α_{in} $(i = 1, 2, 3)$ are the random phase angles determining a certain direction in space therefore the relation $cos^2\alpha_{1n} + cos^2\alpha_{2n} + cos^2\alpha_{3n} = 1$ should be satisfied.

According to our conditions the origin of the ξ, η, ζ coordinate system in the mapped space of turbulence corresponds to an arbitrary point P in the physical space. Consequently, the fluctuation velocity $\mathbf{v}'$ in the moving $q_1^{'}, q_2^{'}, q_3^{'}$ coordinate system can be written as follows:

$$\mathbf{v}'\left(q_1^{'}, q_2^{'}, q_3^{'}, t\right) = \ell\left(q_1^{'}, q_2^{'}, q_3^{'}\right) \Omega\left(q_1^{'}, q_2^{'}, q_3^{'}, t\right)\ \ \nabla \times \mathbf{f}\,(0, 0, 0, \tau)\,. \tag{3.4}$$

Taking equation (3.3) into consideration for the fluctuation velocity we obtain:

$$\mathbf{v}'\left(q_1^{'}, q_2^{'}, q_3^{'}, t\right) = l\Omega\varepsilon \sum_{n=1}^{N} \begin{pmatrix} C_{1n} \cos\left(n\omega\tau + \alpha_{1n}\right) - C_{2n} \sin\left(n\omega\tau + \alpha_{2n}\right) \\ C_{2n} \cos\left(n\omega\tau + \alpha_{2n}\right) - C_{3n} \sin\left(n\omega\tau + \alpha_{3n}\right) \\ C_{3n} \cos\left(n\omega\tau + \alpha_{3n}\right) - C_{1n} \sin\left(n\omega\tau + \alpha_{1n}\right) \end{pmatrix}.$$

The turbulent stress tensor $\mathbf{F}_R$ can be given in the $q_1^{'}, q_2^{'}, q_3^{'}$ coordinate system, considering equation (2.3):

$$\begin{aligned} \mathbf{F}_R &= -\rho\left(\overline{\mathbf{v}' \circ \mathbf{v}'}\right) = -\rho\left(\ell\ \Omega\right)^2 \left(\overline{\nabla\times\mathbf{f}\circ\nabla\times\mathbf{f}}\right) = \\ &= -\rho l^2 \left(\overline{\nabla\times\mathbf{f}\circ\nabla\times\mathbf{f}}\right)\left(\frac{1}{H_1' H_2'}\frac{\partial\left(v_{1'} H_1'\right)}{\partial q_2^{'}}\right)^2 . \end{aligned}$$

Let us introduce the following notations:

$$\alpha = \frac{\overline{(\nabla\times\mathbf{f})_\xi^2}}{\overline{(\nabla\times\mathbf{f})_\xi(\nabla\times\mathbf{f})_\eta}}\ ;\quad \beta = \frac{\overline{(\nabla\times\mathbf{f})_\eta^2}}{\overline{(\nabla\times\mathbf{f})_\xi(\nabla\times\mathbf{f})_\eta}}\ ;\quad \gamma = \frac{\overline{(\nabla\times\mathbf{f})_\zeta^2}}{\overline{(\nabla\times\mathbf{f})_\xi(\nabla\times\mathbf{f})_\eta}}$$

$$\mu = \frac{\overline{(\nabla\times\mathbf{f})_\xi(\nabla\times\mathbf{f})_\zeta}}{\overline{(\nabla\times\mathbf{f})_\xi(\nabla\times\mathbf{f})_\eta}}\ ;\quad \vartheta = \frac{\overline{(\nabla\times\mathbf{f})_\eta(\nabla\times\mathbf{f})_\zeta}}{\overline{(\nabla\times\mathbf{f})_\xi(\nabla\times\mathbf{f})_\eta}}\ ;\quad \kappa^2 = \overline{(\nabla\times\mathbf{f})_\xi(\nabla\times\mathbf{f})_\eta}\,.$$

With these notations the so-called similarity tensor assumes the form

$$\mathbf{H} = \begin{pmatrix} \alpha & 1 & \mu \\ 1 & \beta & \vartheta \\ \mu & \vartheta & \gamma \end{pmatrix}. \tag{3.5}$$

We accept the convention that the sign of the dominant shear stress on the surface with a normal vector perpendicular to the direction of the flow, is the same as that of the derivative of the velocity in the direction of the normal vector.

The turbulent stress tensor $\mathbf{F}_R$ in the $q_1^{'}, q_2^{'}, q_3^{'}$ natural coordinate system is of the form

$$\mathbf{F}_R\left(q_1^{'}, q_2^{'}, q_3^{'}, t\right) = \rho\,\kappa^2\,\ell^2\,\mathbf{H}\left|\frac{1}{H_1' H_2'}\frac{\partial\left(v_{1'} H_1'\right)}{\partial q_2^{'}}\right|\frac{1}{H_1' H_2'}\frac{\partial\left(v_{1'} H_1'\right)}{\partial q_2^{'}}. \tag{3.6}$$

The parameter ε in equation (3.3) is to be chosen in such a way that the *Kármán constant* has the value $\kappa = 0,40704$. The components of the similarity tensor $\mathbf{H}$ in the $q_1^{'}, q_2^{'}, q_3^{'}$ natural coordinate system are constants so they express a kind of proportionality, which is the reason for the name of similarity. Furthermore introducing the notation:

$$\Theta\left(q_1^{'}, q_2^{'}, q_3^{'}, t\right) = \rho\,\kappa^2\,\ell^2\,|\Omega|\ \Omega = \rho\,\kappa^2\,\ell^2\left|\frac{1}{H_1' H_2'}\frac{\partial\left(v_{1'} H_1'\right)}{\partial q_2^{'}}\right|\frac{1}{H_1' H_2'}\frac{\partial\left(v_{1'} H_1'\right)}{\partial q_2^{'}}, \tag{3.7}$$

the turbulent stress tensor $\mathbf{F}_R$ can be written in the natural coordinate system in the following form:

$$\mathbf{F}_R = \Theta\left(q_1^{'}, q_2^{'}, q_3^{'}, t\right)\mathbf{H}, \tag{3.8}$$

where Θ is the element in the first row and second column of the stress tensor, $\Theta \equiv \tau_{12}' = \tau_{21}'$ is the dominant turbulent stress in the flow.

The first step in determining the components of the similarity tensor $\mathbf{H}$ in our example (3.3), the coefficients C_{in} $(i = 1, 2, 3)$ and the phase angles α_{in} $(i = 1, 2, 3)$ must be properly chosen. These quantities are probability variables with uniform distribution functions in the intervals fixed in advance. The next step is to obtain the elements of the turbulent stress tensor $\mathbf{F}_R$, i.e., the time mean values of the components $(\nabla \times \mathbf{f})_j$ $(j = \xi, \eta, \zeta)$ must be determined. These will have the form:

$$\overline{(\nabla \times \mathbf{f})_k (\nabla \times \mathbf{f})_l} = \varepsilon^2 \, U_{k,l} \; ; \quad (k, l = \xi, \eta, \zeta) \; ,$$

in which $U_{k,l}$ are the sums of the products as follows:

$$\begin{aligned} U_{\xi,\eta} &= U_{\eta,\xi} = \frac{1}{2} \sum_{n=1}^{N} \{C_{1n} C_{2n} \cos(\alpha_{1n} - \alpha_{2n})\} + \\ &+ \frac{1}{2} \sum_{n=1}^{N} \{C_{1n} C_{3n} \sin(\alpha_{1n} - \alpha_{3n}) + C_{2n} C_{3n} \cos(\alpha_{2n} - \alpha_{3n})\} \, , \end{aligned}$$

$$\begin{aligned} U_{\eta,\zeta} &= U_{\zeta,\eta} = \frac{1}{2} \sum_{n=1}^{N} \{C_{2n} C_{3n} \cos(\alpha_{2n} - \alpha_{3n})\} + \\ &+ \frac{1}{2} \sum_{n=1}^{N} \{C_{2n} C_{1n} \sin(\alpha_{2n} - \alpha_{1n}) + C_{3n} C_{1n} \cos(\alpha_{3n} - \alpha_{1n})\} \, , \end{aligned}$$

$$\begin{aligned} U_{\zeta,\xi} &= U_{\xi,\zeta} = \frac{1}{2} \sum_{n=1}^{N} \{C_{3n} C_{1n} \cos(\alpha_{3n} - \alpha_{1n})\} + \\ &+ \frac{1}{2} \sum_{n=1}^{N} \{C_{3n} C_{2n} \sin(\alpha_{3n} - \alpha_{2n}) + C_{1n} C_{2n} \cos(\alpha_{1n} - \alpha_{2n})\} \, , \end{aligned}$$

$$U_{\xi,\xi} = \frac{1}{2} \sum_{n=1}^{N} \left\{C_{1n}^2 + C_{2n}^2 + 2 C_{1n} C_{2n} \sin(\alpha_{1n} - \alpha_{2n})\right\} ,$$

$$U_{\eta,\eta} = \frac{1}{2} \sum_{n=1}^{N} \left\{C_{2n}^2 + C_{3n}^2 + 2 C_{2n} C_{3n} \sin(\alpha_{2n} - \alpha_{3n})\right\}$$

$$U_{\zeta,\zeta} = \frac{1}{2} \sum_{n=1}^{N} \left\{C_{3n}^2 + C_{1n}^2 + 2 C_{3n} C_{3n} \sin(\alpha_{3n} - \alpha_{1n})\right\} .$$

The quantities $U_{k,l}$ $(k, l = \xi, \eta, \zeta)$ strictly determine the components of the similarity tensor $\mathbf{H}$ defined by equation (3.5). It is to be mentioned that numerous turbulence models can be created by the selection of the coefficients C_{in}. One of them could be:

$$C_{in} = k_{in} \, exp \left[- \left((n-1) / K \right)^2 \right] .$$

The mean random numbers k_{in} $(i = 1, 2, 3)$ are probability variables having uniform distribution functions in the intervals $[0, \delta_i]$, K is an integer fixed in advance.

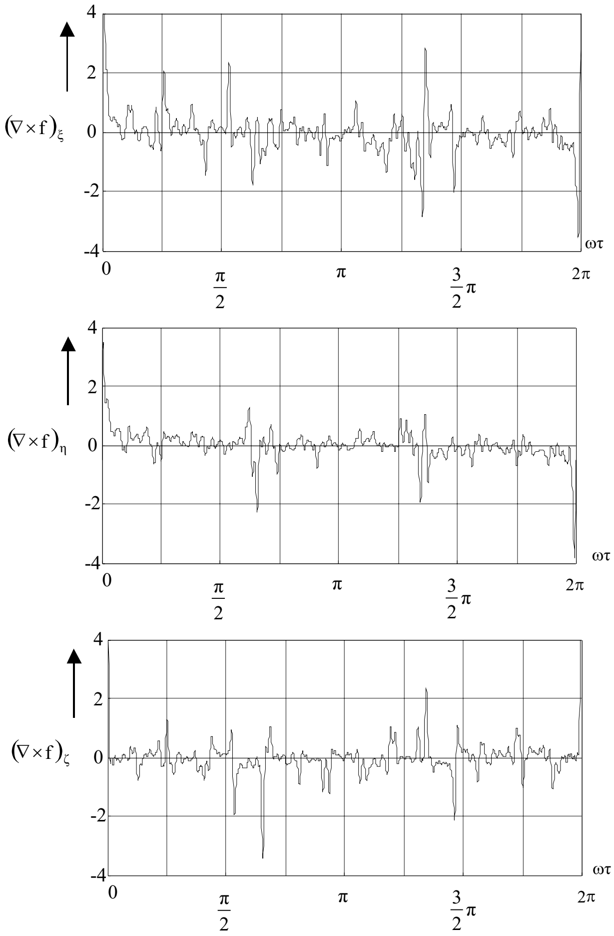

Figure 2. Components of the curl of the vector potential **f** within one period

The variation of the three components of the curl of the dimensionless vector potential can be seen in Figure 2. In this example the data are: $K = 100$; $\delta_1 = 1,0$; $\delta_2 = 0,75$; $\delta_3 = -0,093$, and the components of tensor $\mathbf{H}$ are:

$$\alpha = 3,9714; \ \beta = 1,5734; \ \gamma = 2,8203; \ \mu = 0,9871; \ \vartheta = 0,0002.$$

The *Kármán-constant* $\kappa = 0,40704$ can be obtained by the selection of $\varepsilon = 0,20844$. The α, β, γ values are based on the experimental results of *Laufer* [3] obtained from tests of fully developed turbulent flow in circular pipes. Our aim was simply to show that by the selection of a particular solution of the differential equation (2.6) for the dimensionless vector potential an appropriate stochastic model could be created. Since many particular solutions of the differential equation (2.6) exist, there are various possibilities to create stochastic models fitting different tasks.

The sum of the results is as follows. A natural coordinate system can be attached to the velocity field of the turbulent fluctuation based on the similarity hypothesis. The mean velocity $\mathbf{v}$ and the $\nabla \times \mathbf{v}$ vector determine the basic directions of this system. Moreover, a dimensionless vector potential changing periodically can be associated with this system. The components of the vector potential are composed by a series of waves of random amplitudes and phase angles. The turbulent velocity fluctuation v' can be derived from the vector potential. A 3-D turbulence model is obtained in the way in which the inner mechanism of the turbulence is represented by a stochastic process including 5 independent variables.

4. Turbulence Model in the 3-D Boundary Layer Flow

Along a solid wall in the boundary layer flow below a certain *Reynolds*' number the flow is laminar and it is turbulent above it. The viscosity effect prevails throughout the full layer in case of laminar flow but in the turbulent case only in a thin layer called laminar sublayer. Beyond this sublayer, i.e., in the turbulent boundary layer, the mean characteristic of flow is the turbulent fluctuation causing apparent stress phenomena or in other words apparent friction. The frictionless potential flow is outside of the turbulent layer.

The equations describing the potential flow of an incompressible fluid follow from the equation of continuity and the irrotational velocity field

$$\nabla \cdot \mathbf{v} = 0\ ; \qquad \nabla \times \mathbf{v} = \mathbf{0}.$$

The velocity field can be calculated as the gradient of a velocity potential Φ. Substituting it into the continuity equation, we obtain a differential equation for the velocity potential:

$$\Delta \Phi = 0. \tag{4.1}$$

The boundary conditions for this *Laplace* equation on the solid walls and on the inflow and outflow surfaces respectively of the domain are:

$$(\mathbf{n}\cdot\nabla)\ \Phi = 0 \qquad \text{and} \qquad (\mathbf{n}\cdot\nabla)\ \Phi = v_{B,K}.$$

The unit vector $\mathbf{n}$ is directed outward and normal to the surfaces. The velocity v_B is the distribution normal to the inflow surfaces and v_K is that to the outflow surfaces. There are potential theoretical methods to solve equation (4.1).

The equation of continuity and the momentum equation are decisive in determining the flow of a real fluid. The momentum equation can be replaced by the vortex theorem obtained by taking the curl of the momentum equation. Doing so will increase the order of equations, but the number of unknowns will be less by one. Therefore, it will become easier to set up the closed system of equations for the numerical solution.

We are going to investigate the turbulent flow of incompressible fluid ($\rho = const$) in a force field having potential function ($\mathbf{g} = -\nabla U$). The mean velocity in the q_1, q_2, q_3 orthogonal curvilinear coordinate system is $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$ and the velocity fluctuation is $\mathbf{v}' = v_1'\mathbf{e}_1 + v_2'\mathbf{e}_1 + v_3'\mathbf{e}_3$. The law of mass conservation is expressed by

$$\nabla \cdot \mathbf{v} = 0. \tag{4.2}$$

The *Reynolds*' momentum equation of the turbulent motion of a fluid described by equation (1.7) will be reformulated. The gauge pressure p_R caused by the turbulent velocity fluctuation will be defined by the first scalar invariant of the turbulent stress tensor $\mathbf{F}_R$:

$$p_R = \tfrac{1}{3}\left(\overline{v_1'v_1'} + \overline{v_2'v_2'} + \overline{v_3'v_3'}\right).$$

The deviator $\boldsymbol{\sigma}_R$ of turbulent stress tensor $\mathbf{F}_R$ is created as usual:

$$\boldsymbol{\sigma}_R = -\rho \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\sigma_{11} &= \tfrac{1}{3}\left(2\overline{v_1'v_1'} - \overline{v_2'v_2'} - \overline{v_3'v_3'}\right), \\
\sigma_{22} &= \tfrac{1}{3}\left(2\overline{v_2'v_2'} - \overline{v_3'v_3'} - \overline{v_1'v_1'}\right), \\
\sigma_{33} &= \tfrac{1}{3}\left(2\overline{v_3'v_3'} - \overline{v_1'v_1'} - \overline{v_2'v_2'}\right), \\
\sigma_{12} &= \sigma_{21} = \overline{v_1'v_2'} = \overline{v_2'v_1'}, \\
\sigma_{13} &= \sigma_{31} = \overline{v_1'v_3'} = \overline{v_3'v_1'}, \\
\sigma_{23} &= \sigma_{32} = \overline{v_2'v_3'} = \overline{v_3'v_2'}.
\end{aligned}$$

The *Reynolds*' momentum equation of the turbulent flow of viscid fluid in a conservative force field has the form:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \rho\,\nabla\varPi + \eta\,\triangle\mathbf{v} + \mathrm{Div}\,\boldsymbol{\sigma}_R. \tag{4.3}$$

Here $\varPi = U + p/\rho + p_R/\rho$, the total potential of the incompressible fluid is the sum of the force field potential and the pressure potential. Let $\boldsymbol{\sigma}_R$ be the deviator of the turbulent stress tensor $\mathbf{F}_R$. It follows from equation (3.8) that in the natural coordinate system q_1', q_2', q_3'

$$\boldsymbol{\sigma}_R = \varTheta\left(q_1', q_2', q_3', t\right)\mathbf{H}^*.$$

Here $\mathbf{H}^*$ is the deviator of the similarity tensor $\mathbf{H}$:

$$\mathbf{H}^* = \begin{pmatrix} \alpha_* & 1 & \mu \\ 1 & \beta_* & \vartheta \\ \mu & \vartheta & \gamma_* \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(2\alpha - \beta - \gamma) & 1 & \mu \\ 1 & \frac{1}{3}(2\beta - \gamma - \alpha) & \vartheta \\ \mu & \vartheta & \frac{1}{3}(2\gamma - \alpha - \beta) \end{pmatrix}.$$

Working in an arbitrary q_1, q_2, q_3 orthogonal curvilinear system the deviator tensor $\boldsymbol{\sigma}_R$ should be written in its transformed form:

$$\boldsymbol{\sigma}_R = \varTheta\left(q_1', q_2', q_3', t\right)\mathbf{G}^*. \tag{4.4}$$

The tensor $\mathbf{G}^*$ is the transformed form of the deviator $\mathbf{H}^*$ in the q_1, q_2, q_3 system:

$$\mathbf{G}^* = \mathbf{E} \cdot \mathbf{H}^* \cdot \mathbf{E}^T.$$

The elements of the transformation tensor $\mathbf{E}$ (and its transposed $\mathbf{E}^T$) are the scalar products of the base vectors $E_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j$ $(i, j = 1, 2, 3)$. As the base vectors of the system q'_1, q'_2, q'_3 have previously been given, the elements of $\mathbf{E}$ are as follows:

$$E_{11} = \frac{1}{\sqrt{1-\lambda^2}} \left(\frac{v_1}{v} - \lambda \frac{\Omega_1}{\Omega} \right) \; ; \; E_{12} = \frac{1}{\sqrt{1-\lambda^2}} \frac{v_2 \Omega_3 - v_3 \Omega_2}{v \, \Omega} \; ; \; E_{13} = -\frac{\Omega_1}{\Omega};$$

$$E_{21} = \frac{1}{\sqrt{1-\lambda^2}} \left(\frac{v_2}{v} - \lambda \frac{\Omega_2}{\Omega} \right) \; ; \; E_{22} = \frac{1}{\sqrt{1-\lambda^2}} \frac{v_3 \Omega_1 - v_1 \Omega_3}{v \, \Omega} \; ; \; E_{23} = -\frac{\Omega_2}{\Omega};$$

$$E_{31} = \frac{1}{\sqrt{1-\lambda^2}} \left(\frac{v_3}{v} - \lambda \frac{\Omega_3}{\Omega} \right) \; ; \; E_{32} = \frac{1}{\sqrt{1-\lambda^2}} \frac{v_1 \Omega_2 - v_2 \Omega_1}{v \, \Omega} \; ; \; E_{33} = -\frac{\Omega_3}{\Omega}.$$

Here Ω_i are the scalar components of $\mathbf{\Omega} = \nabla \times \mathbf{v}$ and λ is determined by the scalar product of the unit vectors in the direction velocity and the vortex vector:

$$\begin{aligned}
\Omega_1 &= \frac{1}{H_3 H_2} \left(\frac{\partial (v_3 H_3)}{\partial q_2} - \frac{\partial (v_2 H_2)}{\partial q_3} \right); \\
\Omega_2 &= \frac{1}{H_3 H_1} \left(\frac{\partial (v_1 H_1)}{\partial q_3} - \frac{\partial (v_3 H_3)}{\partial q_1} \right); \\
\Omega_3 &= \frac{1}{H_1 H_2} \left(\frac{\partial (v_2 H_2)}{\partial q_1} - \frac{\partial (v_1 H_1)}{\partial q_2} \right); \\
\lambda &= \frac{v_1 \Omega_1 + v_2 \Omega_2 + v_3 \Omega_3}{\sqrt{v_1^2 + v_2^2 + v_3^2} \, \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}}.
\end{aligned}$$

The elements of $\mathbf{G}^*$ can be calculated easily utilizing the previous relationships, and therefore they may be omitted here.

The *Reynolds*' momentum equation (4.3) with the expression (4.4) for the deviator $\boldsymbol{\sigma}_R$ of the turbulent stress tensor takes the form:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} = \rho \, \nabla \Pi + \eta \, \Delta \mathbf{v} + \text{Div} \, \left(\Theta \mathbf{G}^* \right). \tag{4.5}$$

The motion of real fluid is described by two differential equations namely the *Reynolds*' momentum equation (4.5) and the continuity equation (4.2). Five unknown functions can be found in these four scalar equations $[v_1 (q_1, q_2, q_3, t),\ v_2 (q_1, q_2, q_3, t),$ $v_3 (q_1, q_2, q_3, t),\ \Pi (q_1, q_2, q_3, t)$ and $\Theta (q_1, q_2, q_3, t)]$. Since this system of equations is not closed, one more equation is needed to accomplish numerical calculations. The divergence of equation (4.5) is a possible scalar differential equation for the following reasons. As the fluid is incompressible $\nabla \cdot \mathbf{v} = 0$, while the following vector relationship is valid:

$$\nabla \cdot \left[\left(\mathbf{v} \cdot \boldsymbol{\nabla} \right) \mathbf{v} \right] = \nabla \cdot \left[\left(\mathbf{v} \circ \mathbf{v} \right) \cdot \nabla \right] = \left(\mathbf{v} \circ \nabla \right) : \left(\nabla \circ \mathbf{v} \right).$$

Applying these relationships to equation (4.5) for the total potential Π we obtain the following scalar differential equation:

$$\rho\,\Delta\Pi = -\rho\,(\mathbf{v}\circ\,\nabla) : (\nabla\circ\,\mathbf{v}) + \nabla\cdot\text{Div }\,(\Theta\mathbf{G}^*)\,. \tag{4.6}$$

Here the colon means double scalar products of tensors or dyads. Since in the equation (4.6) a further unknown function does not occur, it will make the differential equation system (4.2) and (4.5) closed.

There is another way to determine the flow of a real fluid numerically apart from the previously mentioned system composed of equations (4.2), (4.5) and (4.6). The vortex theorem (1.9) can be reshaped utilizing the stochastic turbulence model. The time mean value of $\mathbf{v}'\times\mathbf{\Omega}'$ can be determined taking into consideration the fact that $\nabla\cdot\mathbf{v}'=0$ and the following relationship:

$$\mathbf{v}'\times(\nabla\times\mathbf{v}') = \tfrac{1}{2}\nabla\,(\mathbf{v}'\cdot\mathbf{v}') - (\mathbf{v}'\cdot\nabla)\,\mathbf{v}' = \tfrac{1}{2}\nabla\,(\mathbf{v}'\cdot\mathbf{v}') - \text{Div }\,(\mathbf{v}'\circ\mathbf{v}')\,.$$

Thus the time average will be:

$$\overline{\mathbf{v}'\times\mathbf{\Omega}'} = \nabla\frac{\overline{\mathbf{v}'^2}}{2} - \text{Div }\,\left(\overline{\mathbf{v}'\circ\mathbf{v}'}\right) = \nabla\frac{\overline{\mathbf{v}'^2}}{2} + \frac{1}{\rho}\text{Div }\mathbf{F}_R.$$

Since *Reynolds*' stress tensor is $\mathbf{F}_R = -p_R\mathbf{I} + \Theta\mathbf{G}^*$ and $\nabla\times\left(\overline{\mathbf{v}'\times\mathbf{\Omega}'}\right) = \nabla\times$ Div $(\Theta\mathbf{G}^*)$, the vortex theorem (1.9) with the stochastic model of turbulence has the form:

$$\frac{\partial\mathbf{\Omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\,\mathbf{v} = \nu\Delta\mathbf{\Omega} + \frac{1}{\rho}\nabla\times\text{Div }\,(\Theta\mathbf{G}^*)\,. \tag{4.7}$$

This equation consists of three scalar equations, which together with the continuity equation (4.2) compose a closed differential equation system. Namely, there are four unknown functions $[v_1\,(q_1,q_2,q_3,t),\, v_2\,(q_1,q_2,q_3,t),\, v_3\,(q_1,q_2,q_3,t)$ and $\Theta\,(q_1,q_2,q_3,t)]$ and because the components of vortex vector $\mathbf{\Omega}$ can be expressed by the velocity components, the last quantity is not considered as an unknown function.

The sum of this study is as follows. On the basis of the stochastic model of turbulence for the *Reynolds*' stress tensor (and its deviator) a tensor equation can be established in which in addition to the three velocity components there is only one unknown scalar function, namely, the dominant turbulent stress component. With the help of this tensor equation a closed differential equation system can be set up for numerical determination of the turbulent boundary flow of viscid fluid.

REFERENCES

1. Kármán, Th. v.: *Mechanische Ähnlichkeit und Turbulenz* Proc. 3rd Congress of Applied Mechanics 1930, 86-93.
2. Kotschin, N.J. Kiebel, L.A. Rose, N.W: *Theoretische Hydromechanik,* Akademie Verlag Berlin 1954 I, 138-142.
3. Laufer, J: *The Structure of Turbulence in Fully Developed Pipe Flow,* NACA Report 1147, 1954.