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CALCULATION OF THE WORK DONE BY DEFORMATION DEPENDENT TRACTIONS

Imre Kozák
Department of Mechanics, University of Miskolc
3515 Miskolc–Egyetemváros, Hungary
mechkoz@gold.uni-miskolc.hu

Tamás Szabó
Numerical Mechanics Research Group - Hungarian Academy of Sciences
3515 Miskolc–Egyetemváros, Hungary
mechsza@gold.uni-miskolc.hu

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Dedicated to Professor Gyula Béda on the occasion of his seventieth birthday

Abstract. A generalization of the well-known Nanson's formula for solids has been set up for shells. These formulae relate a surface element vector taken either on the base surface or on the side surface in the reference configuration to any surface element vector in the instantaneous configuration. The characteristic quantities can be given by their truncated Taylor expansions with respect to the control parameter. In this way, the work increment done by a deformation dependent traction can always be calculated through integrals taken in the reference configuration, i.e., for shells on the base surface or on the side surface.

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1. **Introduction**

In order to solve a statical problem, calculation of the work increment done by the loads either during an increment in the kinematically admissible displacement or during an increment in the effective displacement is frequently required. This situation arises, for example, in the application of the incremental form of the principle of virtual work (see, e.g., Molk et al [1], Kozák and Szabó [2]). This is also the case in the course of the solution of stability problems when the work increment done by the loads of an equilibrium configuration during a small perturbation in displacements is calculated.

In this paper the applied normal tractions are deformation dependent and are always perpendicular to the instantaneous surface of the body. The aim of this paper is to calculate the work increment done by the loads of any equilibrium configuration through an arbitrary displacement increment, both for an arbitrary body and for shells. Large strains are assumed.

In Section 2 three coordinate systems are introduced: namely, the first is defined in the instantaneous configuration, the second is given in the reference configuration of an arbitrary body and the third is associated with the base surface of the shell also in the reference configuration. The geometrical characteristics of the coordinate systems are also presented together with the applied notations.

In Section 3 relationships are established for the surface element vectors concerning both arbitrary solids and shells in an instantaneous configuration and in a reference one. For shells we distinguish a surface element on the base surface from a surface element on the side surface.

Section 4 gives the forces exerted on surface elements at the points of an instantaneous configuration by means of surface elements at the points of the base surface of a shell.

Section 5 details the formulation of the work increment done by the deformation dependent tractions during an arbitrary increment in the displacements. The characteristic quantities are expanded in truncated Taylor series written in an equilibrium configuration with respect to a control parameter. Then the power of the loads is calculated for each intermediate state of a displacement increment, and finally the work increment is given by integrating the power with respect to the control parameter. As a matter of fact the work increment is given by surface integrals, which are calculated by making use of those quantities defined in the reference configuration. In case of shells the integrals are taken on the base surface or on the side surface.

Invariant and indicial notations are used throughout this paper.

In invariant notation vectors and tensors are denoted by bold lower case letters and upper case letters, respectively. A dot denotes the scalar product, e.g. $\boldsymbol{Q}\cdot \mathrm{d}\mathbf{A}$.

In case of indicial notations the coordinate systems we use are assumed to be curvilinear ones. Latin and Greek indices range over the integers 1,2,3 and 1,2. A subscript preceded by a (comma)[semicolon] denotes (partial)[covariant] differentiation with respect to the corresponding coordinate. δ_k^l is the Kronecker delta while e_{klm} and e^{pqr} stand for the permutation symbols.

2. **Configurations**

2.1. Let us denote the stress and deformation-free reference configuration and the present or instantaneous configuration by (B) and $(\overline{B})$, respectively. Configuration $(\overline{B})$ is an equilibrium one for the actual load level.

In order to extend our investigations for shells, we introduce a base surface (A°) bounded by a closed curve (g°) in the reference configuration (B).

Body forces are neglected, only distributed normal tractions exerted on instantaneous outer surfaces are considered (deformation dependent loads or follower loads).

It is assumed that the kinematical boundary conditions are independent of deformations.

Let us denote the various coordinate systems, a point, the position vector, the local base vectors and the metric tensors in the three different configurations as follows:

– in the current configuration ($\overline{B}$):

$$\{\overline{x}^p\}, \ \overline{P}, \ \overline{\mathbf{r}}, \ \overline{\mathbf{g}}_p, \ \overline{\mathbf{g}}^q, \ \overline{g}_{pq}, \ \overline{g}^{rs},$$

– at the arbitrary point P of the reference configuration (B):

$$\{x^k\}, \ P, \ \mathbf{r}, \ \mathbf{g}_k, \ \mathbf{g}^l, \ g_{kl}, \ g^{mn},$$

– on the base surface (A°) of the shell in the reference configuration (B):

$$\{x^{\circ a}\}, \ P^\circ, \ \mathbf{r}^\circ, \ \mathbf{g}^\circ_a, \ \mathbf{g}^{\circ b}, \ g^\circ_{ab}, \ g^{\circ cd}.$$

REMARK 2.1. The curvilinear coordinate systems introduced are arbitrary except the one defined for shells in the reference configuration (B).

2.2. For shells in the reference configuration (B) (see Figure 1) we have at the point P° of the base surface (A°) for which $x^3 = 0$ that:

$$\mathbf{g}^\circ_\alpha = \frac{\partial \mathbf{r}^\circ}{\partial x^{\circ\alpha}}, \quad \mathbf{g}^\circ_3 = \mathbf{g}^{\circ 3} = \frac{\mathbf{g}^\circ_1 \times \mathbf{g}^\circ_2}{|\mathbf{g}^\circ_1 \times \mathbf{g}^\circ_2|}\,. \tag{2.1}$$

If we regard an arbitrary point P $\left(x^3 \neq 0\right)$ then

$$\mathbf{r} = \mathbf{r}^\circ + \mathbf{g}^\circ_3 x^3, \tag{2.2}$$

$$\mathbf{g}_\alpha = \mathbf{g}^\circ_\alpha - b^{\circ\beta}_\alpha \mathbf{g}^\circ_\beta x^3 = \left(\delta^\beta_\alpha - b^{\circ\beta}_\alpha x^3\right)\mathbf{g}^\circ_\beta = \mu^{\beta^\circ}_\alpha \mathbf{g}^\circ_\beta, \quad \mathbf{g}_3 = \mathbf{g}^\circ_3 \tag{2.3}$$

where $b^{\circ\beta}_\alpha = -\,\mathbf{g}^\circ_{3,\alpha} \cdot \mathbf{g}^{\circ\beta}$ is the tensor of curvature on the surface (A°) and

$$\mu^{\beta^\circ}_\alpha = \delta^\beta_\alpha - b^{\circ\beta}_\alpha x^3, \quad \mu^{\beta^\circ}_3 = 0, \quad \mu^{3^\circ}_3 = 1 \tag{2.4}$$

where $\mu^{b^\circ}_a$ are the coordinates of a shifter. The inverse shifter is denoted by $\mu^d_{c^\circ}$.

REMARK 2.2. The coordinates x^α and $x^{\circ\alpha}$ of the points P and P° are identical for shells $x^\alpha = x^{\circ\alpha}$, however the corresponding coordinates x^3 are different.

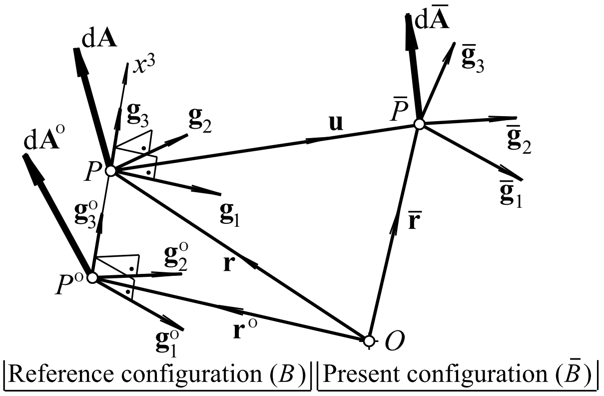
2.3. The permutation tensors taken at the point

$\bar{P}$ are denoted by $\quad \overline{\varepsilon}_{pqr} = \sqrt{\overline{g}}e_{pqr}, \quad \overline{\varepsilon}^{rst} = \dfrac{1}{\sqrt{\overline{g}}}e^{rst}, \qquad \left(\overline{g} = \det \overline{g}_{pq}\right), \qquad (2.5)$

P are denoted by $\quad \varepsilon_{klm} = \sqrt{g}e_{klm}, \quad \varepsilon^{lmn} = \dfrac{1}{\sqrt{g}}e^{lmn}, \qquad (g = \det g_{kl}), \qquad (2.6)$

P° are denoted by $\quad \varepsilon^\circ_{abc} = \sqrt{g^\circ}e_{abc}, \quad \varepsilon^{\circ bcd} = \dfrac{1}{\sqrt{g^\circ}}e^{bcd}. \quad (g^\circ = \det g^\circ_{ab})\,. \qquad (2.7)$

2.4. In accordance with the notations introduced the following conventions are applied. In case of invariant notations a barred letter, a single letter, or a letter with a small circle as a superscript identifies the point from the triplet $\overline{P}$, P or P°, at which the quantity denoted by the letter is defined. In addition to this, when indicial notations are used the same tensor or vector can be written in the local coordinate



Figurre 1. Base vectors and urface element vectors of shells

system of different points, e.g., the surface element vector defined at point $\overline{P}$ can be written as:

$$\mathrm{d}\overline{\mathbf{A}} = \mathrm{d}\overline{A}_{\overline{p}}\overline{\mathbf{g}}^{p} = \mathrm{d}\overline{A}_{k}\mathbf{g}^{k} = \mathrm{d}\overline{A}_{a^\circ}\mathbf{g}^{\circ a}. \tag{2.8}$$

The displacement vector of an arbitrary point can be defined at point $\overline{P}$ and also at point P, and it can be given by the base vectors of point P° as well:

$$\overline{\mathbf{u}} = \overline{u}_{\overline{p}}\overline{\mathbf{g}}^{p} = u_{k}\mathbf{g}^{k} = \mathbf{u} = u_{a^\circ}\mathbf{g}^{\circ a}. \tag{2.9}$$

For our later considerations we shall distinguish the arbitrary coordinate system in the reference configuration (B) from the one defined for shells in the same configuration.

3. **Surface element vectors**

3.1. A geometrical representation. For shells Figure 1 shows the three surface element vectors defined at points $\overline{P}$, P and P°:

$$\overline{P} : \mathrm{d}\overline{\mathbf{A}} = \mathrm{d}\overline{A}_{\overline{p}}\overline{\mathbf{g}}^{p} = \mathrm{d}\overline{\mathbf{r}}_{\mathrm{I}} \times \mathrm{d}\overline{\mathbf{r}}_{\mathrm{II}}, \quad \mathrm{d}\overline{A}_{\overline{p}} = \overline{\varepsilon}_{pqr}\mathrm{d}\overline{x}^{q}_{\mathrm{I}}\mathrm{d}\overline{x}^{r}_{\mathrm{II}}, \tag{3.1}$$

$$P : \mathrm{d}\mathbf{A} = \mathrm{d}A_{k}\mathbf{g}^{k} = \mathrm{d}\mathbf{r}_{\mathrm{I}} \times \mathrm{d}\mathbf{r}_{\mathrm{II}}, \quad \mathrm{d}A_{k} = \varepsilon_{klm}\mathrm{d}x^{l}_{\mathrm{I}}\mathrm{d}x^{m}_{\mathrm{II}}, \tag{3.2}$$

$$P^\circ : \mathrm{d}\mathbf{A}^\circ = \mathrm{d}A^\circ_{a}\mathbf{g}^{\circ a} = \mathrm{d}\mathbf{r}^\circ_{\mathrm{I}} \times \mathrm{d}\mathbf{r}^\circ_{\mathrm{II}}, \quad \mathrm{d}A^\circ_{a^\circ} = \varepsilon^\circ_{abc}\mathrm{d}x^{\circ b}_{\mathrm{I}}\mathrm{d}x^{\circ c}_{\mathrm{II}} \tag{3.3}$$

where $\mathrm{d}\overline{\mathbf{r}}_{\mathrm{I}}$, $\mathrm{d}\mathbf{r}_{\mathrm{I}}$, $\mathrm{d}\mathbf{r}^\circ_{\mathrm{I}}, \ldots$ are line elements at points $\overline{P}$, P and P°, respectively.

REMARK 3.1. The surface element vector $\mathrm{d}\mathbf{A}^\circ$ can be oriented arbitrarily in comparison with the base surface (A°).

Let us set ourselves a task to derive three relationships for an arbitrary body and arbitrary shells between the following pairs of surface element vectors

$$\mathrm{d}\overline{\mathbf{A}} \text{ and } \mathrm{d}\mathbf{A}, \quad \mathrm{d}\mathbf{A} \text{ and } \mathrm{d}\mathbf{A}^\circ, \quad \mathrm{d}\overline{\mathbf{A}} \text{ and } \mathrm{d}\mathbf{A}^\circ.$$

3.2. An arbitrary coordinate system in configuration (B)**.** The mapping between the configurations $(\overline{B})$ and (B) is given either by the function of motion or by the displacement field

$$\overline{x}^p = \overline{x}^p(x^1, x^2, x^3; t), \quad J = \det \frac{\partial \overline{x}^p}{\partial x^k} \neq 0, \tag{3.4}$$

$$\overline{\mathbf{r}} = \mathbf{r} + \mathbf{u}, \quad \mathbf{u} = \mathbf{u}(x^1, x^2, x^3; t). \tag{3.5}$$

For the line element $\mathrm{d}\overline{\mathbf{r}}$ we can write the following relationships:

$$\mathrm{d}\overline{\mathbf{r}} = \overline{\mathbf{g}}_q \mathrm{d}\overline{x}^q = \frac{\partial \overline{\mathbf{r}}}{\partial \overline{x}^q}\mathrm{d}\overline{x}^q = \frac{\partial(\mathbf{r}+\mathbf{u})}{\partial x^l}\frac{\partial x^l}{\partial \overline{x}^q}\mathrm{d}\overline{x}^q = (\delta^l_s + u^l_{\ ;s})\mathrm{d}x^s \mathbf{g}_l, \tag{3.6}$$

where

$$\mathrm{d}x^s = \frac{\partial x^s}{\partial \overline{x}^q}\mathrm{d}\overline{x}^q$$

which follows from the inverse function of motion $x^s = x^s(\overline{x}^1, \overline{x}^2, \overline{x}^3; t)$.

Using equation (3.6) for the line elements $\mathrm{d}\mathbf{r}_{\mathrm{I}}$ and $\mathrm{d}\mathbf{r}_{\mathrm{II}}$, the surface element vector given by equation (3.1) can be written as:

$$\mathrm{d}\overline{\mathbf{A}} = \mathrm{d}\overline{\mathbf{r}}_{\mathrm{I}} \times \mathrm{d}\overline{\mathbf{r}}_{\mathrm{II}} = \mathbf{g}^k \varepsilon_{klm}(\delta^l_s + u^l_{\ ;s})(\delta^m_t + u^m_{\ ;t})\mathrm{d}x^s_{\mathrm{I}}\mathrm{d}x^t_{\mathrm{II}}. \tag{3.7}$$

Making use of the identity

$$\varepsilon_{klm}(\delta^l_s + u^l_{\ ;s})(\delta^m_t + u^m_{\ ;t}) = \frac{1}{2}e_{klm}e^{hij}(\delta^l_i + u^l_{\ ;i})(\delta^m_j + u^m_{\ ;j})\varepsilon_{hst} \tag{3.8}$$

the surface element vector can be expressed as

$$\mathrm{d}\overline{\mathbf{A}} = \mathbf{g}^k Q_k^{\ h}\mathrm{d}A_h, \quad Q_k^{\ h} = \frac{1}{2}e_{klm}e^{hij}(\delta^l_i + u^l_{\ ;i})(\delta^m_j + u^m_{\ ;j}) \tag{3.9}$$

where $\mathrm{d}A_h$ is the surface element at point P given by (3.2).

The surface element vector $\mathrm{d}\overline{\mathbf{A}}$ at the point $\overline{P}$ given by (3.9) can be related to the surface element vector $\mathrm{d}\mathbf{A}$:

$$\mathrm{d}\overline{\mathbf{A}} = \mathrm{d}\overline{A}_{\overline{q}}\overline{\mathbf{g}}^q = \mathrm{d}\overline{A}_k \mathbf{g}^k = \boldsymbol{Q}\cdot \mathrm{d}\mathbf{A}, \quad \boldsymbol{Q} = Q_k^{\ h}\mathbf{g}^k\mathbf{g}_h, \tag{3.10}$$

$$\mathrm{d}\overline{A}_k = Q_k^{\ h}\mathrm{d}A_h. \tag{3.11}$$

REMARK 3.2. Relationships (3.9)-(3.11) are always satisfied.

REMARK 3.3. Relationships (3.9) and (3.10) can be manipulated into a different form by the use of the Nanson formula (see, e.g., Béda et al [3], Mason [4]):

$$\mathrm{d}\overline{\mathbf{A}} = J\sqrt{\frac{\overline{g}}{g}}\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\cdot \mathrm{d}\mathbf{A},$$

where $\boldsymbol{F}^{-1}$ is the inverse of the deformation gradient and T denotes the transpose.

3.3. A coordinate system for shells in configuration (B)**.** For shells one can shift the surface element vector $\mathrm{d}\mathbf{A}$ to point P° by the aid of an inverse shifter $\mu^k_{a^\circ}$

$$\mathrm{d}A_{a^\circ} = \mu^k_{a^\circ}\mathrm{d}A_k. \tag{3.12}$$

In accordance with Remark 2.2. we shall assume at points P° and P that

$$\mathrm{d}x^{\circ b} = \mathrm{d}x^b. \tag{3.13}$$

and using (3.12), (3.2), we have

$$\mathrm{d}A_{a^\circ} = \mu^k_{a^\circ}\varepsilon_{klm}\mathrm{d}x^l_{\mathrm{I}}\mathrm{d}x^m_{\mathrm{II}} = \varepsilon^\circ_{abc}\mu^{b^\circ}_l\mu^{c^\circ}_m\mathrm{d}x^l_{\mathrm{I}}\mathrm{d}x^m_{\mathrm{II}}. \tag{3.14}$$

Introducing the notations

$$e_{abc}\mu^{a^\circ}_k\mu^{b^\circ}_l\mu^{c^\circ}_m = De_{klm}, \qquad D = \det\left|\mu^{a^\circ}_l\right| \tag{3.15}$$

and utilizing equation (3.14) we can write:

$$\mathrm{d}A_k = \mu^{a^\circ}_k\mathrm{d}A_{a^\circ} = \varepsilon^\circ_{abc}\mu^{a^\circ}_k\mu^{b^\circ}_l\mu^{c^\circ}_m\mathrm{d}x^l_{\mathrm{I}}\mathrm{d}x^m_{\mathrm{II}} = D\mathrm{d}A^\circ_{k^\circ}, \tag{3.16}$$

where with regard to (3.3) and (3.13)

$$\mathrm{d}A^\circ_{k^\circ} = \varepsilon^\circ_{klm}\mathrm{d}x^l_{\mathrm{I}}\mathrm{d}x^m_{\mathrm{II}}. \tag{3.17}$$

As a consequence the surface element vector $\mathrm{d}\mathbf{A}$ at point P can be given with the aid of $\mathrm{d}\mathbf{A}^\circ$ at point P°. With a view to (3.8) and (3.16), it follows from (3.14) that:

$$\mathrm{d}\mathbf{A} = \mathrm{d}A_k\mathbf{g}^k = \mathrm{d}A_{a^\circ}\mathbf{g}^{\circ a} = \boldsymbol{W}^\circ\cdot\mathrm{d}\mathbf{A}^\circ, \quad \boldsymbol{W}^\circ = W^{\circ\ h^\circ}_{a^\circ}\mathbf{g}^{\circ a}\mathbf{g}^\circ_h, \tag{3.18}$$

$$\mathrm{d}A_{a^\circ} = W^{\circ\ h^\circ}_{a^\circ}\mathrm{d}A^\circ_{h^\circ}, \qquad W^{\circ\ h^\circ}_{a^\circ} = \frac{1}{2}e_{abc}e^{hij}\mu^{b^\circ}_i\mu^{c^\circ}_j, \tag{3.19}$$

$$\mathrm{d}A_k = D\mathrm{d}A^\circ_{k^\circ}. \tag{3.20}$$

REMARK 3.4. Relationships (3.18)-(3.20) are valid independently of the orientation of $\mathrm{d}\mathbf{A}^\circ$.

Derivation of the third relationship we planned to set up at the beginning of Section 3 can be carried out as follows. Writing equation (3.7) as

$$\mathrm{d}\overline{\mathbf{A}} = \mathbf{g}^{\circ a}\varepsilon^\circ_{abc}\mu^{b^\circ}_l(\delta^l_s + u^l_{\ ;s})\mu^{c^\circ}_m(\delta^m_t + u^m_{\ ;t})\mathrm{d}x^s_{\mathrm{I}}\mathrm{d}x^t_{\mathrm{II}}$$

and introducing the notations

$$\mathbf{u} = u^l\mathbf{g}_l = u^{b^\circ}\mathbf{g}^\circ_b, \quad \mu^{b^\circ}_l u^l_{\ ;s} = u^{b^\circ}_{\ ;s} \tag{3.21}$$

$$u^{b^\circ}_{\ ;\alpha} = u^{b^\circ}_{\ ,\alpha} + \Gamma^{b^\circ}_{\alpha^\circ n^\circ}u^{n^\circ}, \quad u^{b^\circ}_{\ ;3} = u^{b^\circ}_{\ ,3} \tag{3.22}$$

we obtain, with a view to (3.8), the following relation:

$$\mathrm{d}\overline{\mathbf{A}} = \mathrm{d}\overline{A}_{a^\circ}\mathbf{g}^{\circ a} = \mathbf{g}^{\circ a}Q^{\circ\ h^\circ}_{a^\circ}\mathrm{d}A^\circ_{h^\circ}, \quad Q^{\circ\ h^\circ}_{a^\circ} = \frac{1}{2}e_{abc}e^{hij}(\mu^{b^\circ}_i + u^{b^\circ}_{\ ;i})(\mu^{c^\circ}_j + u^{c^\circ}_{\ ;j}). \tag{3.23}$$

where $\mathrm{d}A^\circ_{h^\circ}$ can be calculated from (3.17).

Finally we have a formula which relates $\mathrm{d}\overline{\mathbf{A}}$ taken at point $\overline{P}$ to $\mathrm{d}\mathbf{A}^\circ$ taken at point P°:

$$\mathrm{d}\overline{\mathbf{A}} = \mathrm{d}\overline{A}_{\overline{q}}\overline{\mathbf{g}}^q = \mathrm{d}\overline{A}_k\mathbf{g}^k = \mathrm{d}\overline{A}_{a^\circ}\mathbf{g}^{\circ a} = \boldsymbol{Q}^\circ\cdot\mathrm{d}\mathbf{A}^\circ, \quad \boldsymbol{Q}^\circ = Q^{\circ\ h^\circ}_{a^\circ}\mathbf{g}^{\circ a}\mathbf{g}^\circ_h, \tag{3.24}$$

$$\mathrm{d}\overline{A}_{a^\circ} = Q^{\circ\ h^\circ}_{a^\circ}\mathrm{d}A^\circ_{h^\circ}. \tag{3.25}$$

Then combining (3.11) and (3.20) we obtain:

$$\mathrm{d}\overline{A}_k = Q^{\ h}_k\mathrm{d}A_h = DQ^{\ h}_k\mathrm{d}A^\circ_{h^\circ}. \tag{3.26}$$

For shells we distinguish two special cases for the location of the surface element $\mathrm{d}\mathbf{A}^\circ$. In the first case the surface element $\mathrm{d}\mathbf{A}^\circ$ is on the base surface (A°) and in accordance with equation (3.3) we get

$$\mathrm{d}\mathbf{A}^\circ = \mathrm{d}A^\circ_{3^\circ}\mathbf{g}^{\circ 3} = \varepsilon^\circ_{3\sigma\tau}\mathrm{d}x^\sigma_{\mathrm{I}}\mathrm{d}x^\tau_{\mathrm{II}}\mathbf{g}^{\circ 3} \tag{3.27}$$

and

$$\mathrm{d}\overline{\mathbf{A}} = \mathrm{d}\overline{A}_{a^\circ}\mathbf{g}^{\circ a}, \qquad \mathrm{d}\overline{A}_{a^\circ} = Q^{\circ\ 3^\circ}_{a^\circ}\mathrm{d}A^\circ_{3^\circ} = \frac{1}{2}e_{abc}e^{3\eta\vartheta}(\mu^{b^\circ}_{\eta} + u^{b^\circ}_{;\eta})(\mu^{c^\circ}_{\vartheta} + u^{c^\circ}_{;\vartheta})\mathrm{d}A^\circ_{3^\circ}. \tag{3.28}$$

In the second case the surface elements $\mathrm{d}\overline{\mathbf{A}}$, $\mathrm{d}\mathbf{A}$ and $\mathrm{d}\mathbf{A}^\circ$ are situated on the side surface $\left(\overline{A}^*\right)$, or (A^*). Let us denote the surface elements by $\mathrm{d}\overline{\mathbf{A}}^*$, $\mathrm{d}\mathbf{A}^*$ and $\mathrm{d}\mathbf{A}^{\circ *}$ and the corresponding points by $\overline{P}^*$, P^* and $P^{\circ *}$, respectively. (A^*) is determined by the normal vector of the base surface (A°) on the boundary curve (g°). Assume that $x^\vartheta = x^\vartheta(s^\circ)$ is the equation of the boundary curve (g°) on the base surface (A°). Then we can write the unit tangent and the unit normal to the surface (A^*) as

$$\mathbf{t}^\circ = \frac{\mathrm{d}\mathbf{r}^\circ}{\mathrm{d}s^\circ} = \frac{\partial \mathbf{r}^\circ}{\partial x^\vartheta}\frac{\mathrm{d}x^\vartheta}{\mathrm{d}s^\circ} = t^{\circ\vartheta}\mathbf{g}^\circ_\vartheta, \quad \mathbf{n}^\circ = \mathbf{t}^\circ \times \mathbf{g}^\circ_3 = \varepsilon^\circ_{\eta\vartheta 3}t^{\circ\vartheta}\mathbf{g}^{\circ\eta} = n^\circ_\eta\mathbf{g}^{\circ\eta}. \tag{3.29}$$

The surface element at point P° of the curve (g°) on the side surface (A^*) is given by

$$\mathrm{d}\mathbf{A}^{\circ *} = (\mathbf{t}^\circ\mathrm{d}s^\circ) \times \left(\mathbf{g}^\circ_3\mathrm{d}x^3\right) = \mathbf{n}^\circ\mathrm{d}s^\circ\mathrm{d}x^3 = \mathrm{d}A^{\circ *}_{\eta^\circ}\mathbf{g}^{\circ\eta} \tag{3.30}$$

$$\mathrm{d}A^{\circ *}_{\eta^\circ} = n^\circ_{\eta^\circ}\mathrm{d}s^\circ\mathrm{d}x^3 = \varepsilon^\circ_{\eta\vartheta 3}t^{\circ\vartheta}\mathrm{d}s^\circ\mathrm{d}x^3. \tag{3.31}$$

Similarly, at an arbitrary point $\overline{P}^*$ of the side surface $\left(\overline{A}^*\right)$ the surface element can be obtained from (3.25) and (3.26) and can be written as

$$\mathrm{d}\overline{A}^*_{a^\circ} = Q^{\circ *\eta^\circ}_{a^\circ}\mathrm{d}A^{\circ *}_{\eta^\circ}, \qquad \mathrm{d}\overline{A}^*_k = D^*Q^{*\eta}_k\mathrm{d}A^{\circ *}_{\eta^\circ}. \tag{3.32}$$

4. **Deformation dependent tractions**

4.1. An arbitrary coordinate system in configuration (B)**.** We define the traction on the surface part $\left(\overline{A}_t\right)$ of an arbitrary present configuration $\left(\overline{B}\right)$ as

$$\widetilde{p} = p\widetilde{p}_\circ, \quad x \in \left(\overline{A}_t\right), \tag{4.1}$$

where p is the load parameter and $\widetilde{p}_\circ$ is a reference traction regarded as positive if the traction points out of the surface.

The surface part $\left(\overline{A}_t\right)$ corresponds to (A_t) in configuration (B).

In accordance with (3.10) the force acting on the corresponding surface element is given by the formulae:

$$\mathrm{d}\overline{\mathbf{F}} = \widetilde{p}\mathrm{d}\overline{\mathbf{A}} = \widetilde{p}\boldsymbol{Q}\cdot\mathrm{d}\mathbf{A}\,, \tag{4.2}$$

$$\mathrm{d}\overline{F}_{\overline{q}} = \widetilde{p}\mathrm{d}\overline{A}_{\overline{q}}, \quad \mathrm{d}\overline{F}_k = \widetilde{p}\mathrm{d}\overline{A}_k = \widetilde{p}Q^{\ p}_k\mathrm{d}A_p\,. \tag{4.3}$$

REMARK 4.1. Tractions can be exerted on two or more surface parts simultaneously.

4.2. A coordinate system for shells in configuration (B)**.** The shell in configuration (B) is bounded by a top surface (A^+) and a bottom surface (A^-), which are given by the equations $b^+ = b^+\left(x^1, x^2\right)$, $b^- = b^-\left(x^1, x^2\right)$, and a side surface (A^*). The values of b^+ and b^- are measured along the normal vector $\mathbf{g}_3^\circ$ of the base surface (A°): $b^+ \geq x^3 \geq b^-$. The thickness of the shell is $b = b^+ - b^-$.

In the present configuration $(\overline{B})$ the surfaces $\left(\overline{A}^+\right)$, $\left(\overline{A}^-\right)$ and $\left(\overline{A}^*\right)$ correspond to the top surface (A^+), bottom surface (A^-) and side surface (A^*).

We define the tractions on the surface parts $\left(\overline{A}_t^+\right)$ and $\left(\overline{A}_t^*\right)$ of an arbitrary present configuration $(\overline{B})$ of the shell by

$$\widetilde{p}^+ = p^+ \widetilde{p}_\circ^+, \quad x \in \left(\overline{A}_t^+\right), \tag{4.4}$$

$$\widetilde{p}^* = p^* \widetilde{p}_\circ^*, \quad x \in \left(\overline{A}_t^*\right), \tag{4.5}$$

where p^+ and p^* are load parameters, and $\widetilde{p}_\circ^+$ and $\widetilde{p}_\circ^*$ are reference tractions regarded as positive, if the traction is directed out of the surface.

REMARK 4.2. Loads can be exerted on the bottom surface (A^-) as well.

In accordance with equations (3.10), (3.11) and (3.22), (3.23), the forces acting on the corresponding surface elements are given by the formulae:

$$\mathrm{d}\overline{\mathbf{F}}^+ = \widetilde{p}^+ \mathrm{d}\overline{\mathbf{A}}^+ = \widetilde{p}^+ \boldsymbol{Q}^+ \cdot \mathrm{d}\mathbf{A}^+ = \widetilde{p}^+ \boldsymbol{Q}^{\circ +} \cdot \mathrm{d}\mathbf{A}^{\circ +}, \tag{4.6}$$

$$\mathrm{d}\overline{F}_{\overline{q}}^+ = \widetilde{p}^+ \mathrm{d}\overline{A}_{\overline{q}}^+, \quad \mathrm{d}\overline{F}_k^+ = \widetilde{p}^+ \mathrm{d}\overline{A}_k^+ = \widetilde{p}^+ Q_k^{+\,p} \mathrm{d}A_k^+, \quad \mathrm{d}\overline{F}_{a^\circ}^+ = \widetilde{p}^+ \mathrm{d}\overline{A}_{a^\circ}^+ = \widetilde{p}^+ Q_{a^\circ}^{\circ +\,h^\circ} \mathrm{d}A_{h^\circ}^\circ \tag{4.7}$$

$$\mathrm{d}\overline{\mathbf{F}}^* = \widetilde{p}^* \mathrm{d}\overline{\mathbf{A}}^* = \widetilde{p}^* \boldsymbol{Q}^* \cdot \mathrm{d}\mathbf{A}^*, \tag{4.8}$$

$$\mathrm{d}\overline{F}_{\overline{q}}^* = \widetilde{p}^* \mathrm{d}\overline{A}_{\overline{q}}^*, \quad \mathrm{d}\overline{F}_k^* = \widetilde{p}^* \mathrm{d}\overline{A}_k^* = \widetilde{p}^* Q_k^{*\,h} \mathrm{d}A_h^*. \tag{4.9}$$

5. Calculation of the work increment

5.1. We apply the Lagrangian formulation in the reference configuration (B).

Let us introduce a *control parameter* τ and its increment $\Delta\tau$ to describe a small change of an equilibrium configuration $(\overline{B})$ in the interval $0 \leq \tau \leq \Delta\tau$. The control parameter can be either a load parameter (e.g. p, p^+, p^*) or a displacement parameter.

We assume on the initiation made by Marcinowsky [5] that the variables in a small neighborhood of the equilibrium configuration $(\overline{B})$ can be given by truncated Taylor expansions with respect to the control parameter (asymptotic numerical method).

The control parameter is regarded as quasi-time. In our problem

$$u^k(\tau) = u^k_{\mathrm{B}} + \Delta u^k_{\mathrm{B}} = u^k_{\mathrm{B}} + \dot{u}^k_{\mathrm{B}}\tau + \frac{1}{2}\ddot{u}^k_{\mathrm{B}}\tau^2 + \frac{1}{6}\dddot{u}^k_{\mathrm{B}}\tau^3 + \cdots, \tag{5.1}$$

$$p(\tau) = p_{\mathrm{B}} + \Delta p_{\mathrm{B}} = p_{\mathrm{B}} + \dot{p}_{\mathrm{B}}\tau + \frac{1}{2}\ddot{p}_{\mathrm{B}}\tau^2 + \frac{1}{6}\dddot{p}_{\mathrm{B}}\tau^3 + \cdots, \tag{5.2}$$

$$Q^{\,p}_{k}(\tau) = Q_{\mathrm{B}k}{}^{h} + \Delta Q_{\mathrm{B}k}{}^{h} = Q_{\mathrm{B}k}{}^{h} + \dot{Q}_{\mathrm{B}k}{}^{h}\tau + \frac{1}{2}\ddot{Q}_{\mathrm{B}k}{}^{h}\tau^2 + \frac{1}{6}\dddot{Q}_{\mathrm{B}k}{}^{h}\tau^3 + \cdots, \tag{5.3}$$

where a dot (or dots) above a variable denotes differentiation with respect to a control parameter, $\Delta\tau$ belongs to the displacement increment Δu^k_{B}, which determines the configuration $\left(\overline{B} + \Delta\overline{B}\right)$. The subscript B means that the quantity in question is defined in the equilibrium configuration $\left(\overline{B}\right)$.

In accordance with equation (3.9) we can write

$$Q_{\mathrm{B}k}{}^{p} = \frac{1}{2}e_{klm}e^{pqr}(\delta^l_q + u_{\mathrm{B}}{}^{l}_{;q})(\delta^m_r + u_{\mathrm{B}}{}^{m}_{;r}), \tag{5.4}$$

$$\dot{Q}_{\mathrm{B}k}{}^{p} = e_{klm}e^{pqr}(\delta^l_q + u_{\mathrm{B}}{}^{l}_{;q})\dot{u}_{\mathrm{B}}{}^{m}_{;r}, \tag{5.5}$$

$$\ddot{Q}_{\mathrm{B}k}{}^{p} = e_{klm}e^{pqr}\left[(\delta^l_q + u_{\mathrm{B}}{}^{l}_{;q})\ddot{u}_{\mathrm{B}}{}^{m}_{;r} + \dot{u}_{\mathrm{B}}{}^{l}_{;q}\dot{u}_{\mathrm{B}}{}^{m}_{;r}\right], \tag{5.6}$$

$$\dddot{Q}_{\mathrm{B}k}{}^{p} = e_{klm}e^{pqr}\left[(\delta^l_q + u_{\mathrm{B}}{}^{l}_{;q})\dddot{u}_{\mathrm{B}}{}^{m}_{;r} + 3\ddot{u}_{\mathrm{B}}{}^{l}_{;q}\dot{u}_{\mathrm{B}}{}^{m}_{;r}\right]. \tag{5.7}$$

5.2. An arbitrary coordinate system in configuration (B)**.** Using formulae (4.2), (4.3) and (5.1), (5.3) for the power of the traction (4.1) in the equilibrium configuration $\left(\overline{B}\right)$ and in the interval $0 \le \tau \le \Delta\tau$ we can writte

$$\begin{aligned} P(\tau) &= p_{\mathrm{B}}\int_{(A_t)}\widetilde{p}_{\circ}\dot{\mathbf{u}}(\tau)\cdot\mathrm{d}\overline{\mathbf{A}} = P(\tau) = p_{\mathrm{B}}\int_{(A_t)}\widetilde{p}_{\circ}\dot{\mathbf{u}}(\tau)\cdot\mathrm{d}\overline{\mathbf{A}} = \\ &= p_{\mathrm{B}}\int_{(A_t)}\widetilde{p}_{\circ}\dot{\mathbf{u}}(\tau)\cdot\boldsymbol{Q}(\tau)\cdot\mathrm{d}\mathbf{A} = p_{\mathrm{B}}\int_{(A_t)}\widetilde{p}_{\circ}\dot{u}^k(\tau)\,Q^{\,p}_{k}(\tau)\,\mathrm{d}A_p = \\ &= p_{\mathrm{B}}\int_{(A_t)}\widetilde{p}_{\circ}\left(\dot{u}^k_{\mathrm{B}} + \ddot{u}^k_{\mathrm{B}}\tau + \frac{1}{2}\dddot{u}^k_{\mathrm{B}}\tau^2 + \cdots\right)\left(Q_{\mathrm{B}k}{}^{p} + \dot{Q}_{\mathrm{B}k}{}^{p}\tau + \frac{1}{2}\ddot{Q}_{\mathrm{B}k}{}^{p}\tau^2 + \cdots\right)\mathrm{d}A_p. \end{aligned} \tag{5.8}$$

The work increment of the traction in interval $0 \le \tau \le \Delta\tau$ can be determined by integrating the power $P(\tau)$ with respect to τ:

$$\begin{aligned} \Delta W &= \int_{\tau=0}^{\Delta\tau} P(\tau)\,\mathrm{d}\tau = \\ &= p_{\mathrm{B}}\left[\int_{(A_t)}\widetilde{p}_{\circ}\dot{u}^k_{\mathrm{B}}Q_{\mathrm{B}k}{}^{p}\mathrm{d}A_p\right]\Delta\tau + \\ &+ p_{\mathrm{B}}\left[\int_{(A_t)}\widetilde{p}_{\circ}\left(\dot{u}^k_{\mathrm{B}}\dot{Q}_{\mathrm{B}k}{}^{p} + \ddot{u}^k_{\mathrm{B}}Q_{\mathrm{B}k}{}^{p}\right)\mathrm{d}A_p\right]\frac{1}{2}(\Delta\tau)^2 + \\ &+ p_{\mathrm{B}}\left[\int_{(A_t)}\widetilde{p}_{\circ}\left(\dot{u}^k_{\mathrm{B}}\ddot{Q}_{\mathrm{B}k}{}^{p} + 2\ddot{u}^k_{\mathrm{B}}\dot{Q}_{\mathrm{B}k}{}^{p} + \dddot{u}^k_{\mathrm{B}}Q_{\mathrm{B}k}{}^{p}\right)\mathrm{d}A_p\right]\frac{1}{6}(\Delta\tau)^3 + \cdots. \end{aligned} \tag{5.9}$$

If one applies a finite element discretization ΔW can be given – in a view of (5.4)-(5.7) and (5.9) – in terms of $\underline{\mathbf{t}}_{\mathrm{B}}$ and its derivatives $\dot{\underline{\mathbf{t}}}_{\mathrm{B}}$, $\ddot{\underline{\mathbf{t}}}_{\mathrm{B}}$, $\dddot{\underline{\mathbf{t}}}_{\mathrm{B}}, \cdots$ taken with respect to the displacement parameter $\underline{\mathbf{t}}$.

5.3. A coordinate system for shells in configuration (B)**.** Geometrically nonlinear shell theories differ from each other mainly in the applied kinematic assumptions (see, e.g., Basar and Ding [6], Parisch [7], Sansour and Kollmann [8]). In this paper the analysis of the geometrically nonlinear shells is out of scope, therefore we adopt a displacement field without reasoning. Let the displacement be given at an arbitrary point P of the configuration (B) as

$$
\begin{gathered}
\mathbf{u} = u^k \mathbf{g}_k = u^{a^\circ} \mathbf{g}^\circ_a, \\
\mathbf{u} = \mathbf{v}^\circ + \mathbf{w}^\circ x^3 + \mathbf{q}^\circ \left(x^3\right)^2 + \mathbf{s}^\circ \left(x^3\right)^3, \\
u^{a^\circ} = v^{\circ a^\circ} + w^{\circ a^\circ} x^3 + q^{\circ a^\circ} \left(x^3\right)^2 + s^{\circ a^\circ} \left(x^3\right)^3
\end{gathered}
\tag{5.10}
$$

where the functions $\mathbf{v}^\circ$, $\mathbf{w}^\circ$, $\mathbf{q}^\circ$ and $\mathbf{s}^\circ$ are defined on the base surface (A°). With a view to (5.1) and (5.3) we can write

$$
u^{a^\circ}(\tau) = u^{a^\circ}_{\mathrm{B}} + \Delta u^{a^\circ}_{\mathrm{B}} = u^{a^\circ}_{\mathrm{B}} + \dot{u}^{a^\circ}_{\mathrm{B}} \tau + \frac{1}{2} \ddot{u}^{a^\circ}_{\mathrm{B}} \tau^2 + \frac{1}{6} \dddot{u}^{a^\circ}_{\mathrm{B}} \tau^3 + \cdots, \tag{5.11}
$$

$$
Q^{\circ\ h^\circ}_{a^\circ}(\tau) = Q^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} + \Delta Q^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} = Q^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} + \dot{Q}^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} \tau + \frac{1}{2} \ddot{Q}^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} \tau^2 + \frac{1}{6} \dddot{Q}^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} \tau^3 + \cdots \tag{5.12}
$$

and in accordance with (3.23) it follows

$$
Q^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} = \frac{1}{2} e_{abc} e^{hij} (\mu_{\mathrm{B}i}^{\ \ b^\circ} + u_{\mathrm{B}\ \ ;i}^{\ \ b^\circ})(\mu_{\mathrm{B}j}^{\ \ c^\circ} + u_{\mathrm{B}\ \ ;j}^{\ \ c^\circ}), \tag{5.13}
$$

$$
\dot{Q}^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} = e_{abc} e^{hij} (\mu_{\mathrm{B}i}^{\ \ b^\circ} + u_{\mathrm{B}\ \ ;i}^{\ \ b^\circ}) \dot{u}_{\mathrm{B}\ \ ;j}^{\ \ c^\circ}, \tag{5.14}
$$

$$
\ddot{Q}^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} = e_{abc} e^{hij} \left[(\mu_{\mathrm{B}i}^{\ \ b^\circ} + u_{\mathrm{B}\ \ ;i}^{\ \ b^\circ}) \ddot{u}_{\mathrm{B}\ \ ;j}^{\ \ c^\circ} + \dot{u}_{\mathrm{B}\ \ ;i}^{\ \ b^\circ} \dot{u}_{\mathrm{B}\ \ ;j}^{\ \ c^\circ} \right], \tag{5.15}
$$

$$
\dddot{Q}^{\circ\ \ h^\circ}_{\mathrm{B}a^\circ} = e_{abc} e^{hij} \left[(\mu_{\mathrm{B}i}^{\ \ b^\circ} + u_{\mathrm{B}\ \ ;i}^{\ \ b^\circ}) \dddot{u}_{\mathrm{B}\ \ ;j}^{\ \ c^\circ} + 3 \ddot{u}_{\mathrm{B}\ \ ;i}^{\ \ b^\circ} \dot{u}_{\mathrm{B}\ \ ;j}^{\ \ c^\circ} \right], \tag{5.16}
$$

where, e.g.,

$$
\dot{u}^{a^\circ}_{\mathrm{B}} = \dot{v}^{\circ a^\circ}_{\mathrm{B}} + \dot{w}^{\circ a^\circ}_{\mathrm{B}} x^3 + \dot{q}^{\circ a^\circ}_{\mathrm{B}} \left(x^3\right)^2 + \dot{s}^{\circ a^\circ}_{\mathrm{B}} \left(x^3\right)^3. \tag{5.17}
$$

5.3.1. The loaded surfaces $\left(\overline{A}^+_t\right)$, $\left(A^+_t\right)$ and $\left(A^{\circ +}_t\right)$ belong together. Making use of equations (4.7) and (5.11), (5.12) for the power of the traction (4.4) acting on the surface part $\left(\overline{A}^+_t\right)$ in the equilibrium configuration $\left(\overline{B}\right)$ and in the interval

$0 \le \tau \le \Delta\tau$ we can write

$$P(\tau) = p_{\rm B}^{+} \int_{(\bar{A}_t^{+})} \widetilde{p}_\circ^{+} \dot{\mathbf{u}}^{+}(\tau) \cdot \mathrm{d}\overline{\mathbf{A}} = p_{\rm B}^{+} \int_{(A_t^{\circ +})} \widetilde{p}_\circ^{+} \dot{u}^{+a^\circ}(\tau)\, Q_{a^\circ}^{\circ + h^\circ}(\tau)\, \mathrm{d}A_{h^\circ}^{\circ} =$$

$$= p_{\rm B}^{+} \int_{(A_t^{\circ +})} \widetilde{p}_\circ^{+} \left(\dot{u}_{\rm B}^{+a^\circ} + \ddot{u}_{\rm B}^{+a^\circ}\tau + \frac{1}{2}\dddot{u}_{\rm B}^{+a^\circ}\tau^2 + \cdots \right) \cdot$$
$$\cdot \left(Q_{{\rm B}a^\circ}^{\circ +\; h^\circ} + \dot{Q}_{{\rm B}a^\circ}^{\circ +\; h^\circ}\tau + \frac{1}{2}\ddot{Q}_{{\rm B}a^\circ}^{\circ +\; h^\circ}\tau^2 + \cdots \right) \mathrm{d}A_{h^\circ}^{\circ}, \tag{5.18}$$

where in accordance with (5.17), e.g.,

$$\dot{u}_{\rm B}^{+a^\circ} = \dot{v}_{\rm B}^{\circ a^\circ} + \dot{w}_{\rm B}^{\circ a^\circ} b^{+} + \dot{q}_{\rm B}^{\circ a^\circ} \left(b^{+}\right)^2 + \dot{s}_{\rm B}^{\circ a^\circ} \left(b^{+}\right)^3, \tag{5.19}$$

and with a view to (5.14), (2.4) and (5.10), (5.17) we have e.g.,

$$\dot{Q}_{{\rm B}a^\circ}^{\circ +\; h^\circ} = e_{abc} e^{hij} \left\{ \left[\delta_i^b - b_i^{\circ b} b^{+} + v_{{\rm B}\;;i}^{\circ b^\circ} + w_{{\rm B}\;;i}^{\circ b^\circ} b^{+} + q_{{\rm B}\;;i}^{\circ b^\circ} \left(b^{+}\right)^2 + s_{{\rm B}\;;i}^{\circ b^\circ} \left(b^{+}\right)^3 \right] \cdot \right.$$
$$\left. \cdot \left[\dot{v}_{{\rm B}\;;j}^{\circ c^\circ} + \dot{w}_{{\rm B}\;;j}^{\circ c^\circ} b^{+} + \dot{q}_{{\rm B}\;;j}^{\circ c^\circ} \left(b^{+}\right)^2 + \dot{s}_{{\rm B}\;;j}^{\circ c^\circ} \left(b^{+}\right)^3 \right] \right\}. \tag{5.20}$$

The work increment of the traction in the interval $0 \le \tau \le \Delta\tau$ can be obtained by integrating the power $P(\tau)$ with respect to τ:

$$\Delta W = \int_{\tau=0}^{\Delta\tau} P(\tau)\, \mathrm{d}\tau. \tag{5.21}$$

This formula can also be detailed if we take equations (5.10) and (5.13)-(5.16) into account.

5.3.2. In the special case when the side surface $\left(\overline{A}^{*}\right)$ is subjected to tractions we can also set up the required relationships. Using formula (3.25) for the power of the traction (4.5) exerted on the surface part $\left(\overline{A}_t^{*}\right)$ in the equilibrium configuration $(\overline{B})$ and in the interval $0 \le \tau \le \Delta\tau$ we obtain

$$P(\tau) = p_{\rm B}^{*} \int_{(\bar{A}_t^{*})} \widetilde{p}_\circ^{*} \dot{\mathbf{u}}^{*}(\tau) \cdot \mathrm{d}\overline{\mathbf{A}}^{*} = p_{\rm B}^{*} \int_{(A_t^{*})} \widetilde{p}_\circ^{*} \dot{u}^{*a^\circ}(\tau)\, Q_{a^\circ}^{\circ * \eta^\circ}(\tau)\, \mathrm{d}A_{\eta^\circ}^{\circ *}.$$

In view of (3.31), the surface integral can be decomposed into two line integrals. Integration is performed in two steps, first along the thickness and then on the part (g_t°) of the boundary curve (g°) – this part belongs to (A_t^{*}):

$$P(\tau) = p_{\rm B}^{*} \int_{(g_t^\circ)} \left[\int_{(b)} \widetilde{p}_\circ^{*} \dot{u}^{*a^\circ}(\tau)\, Q_{a^\circ}^{\circ * h^\circ}(\tau)\, \mathrm{d}x^3 \right] \varepsilon_{\eta\vartheta 3}^{\circ} t^{\circ\vartheta} \mathrm{d}s^\circ. \tag{5.22}$$

The work increment ΔW is obtained by integrating the power $P(\tau)$ with respect to τ in the interval $0 \leq \tau \leq \Delta\tau$:

$$\Delta W = \int_{\tau=0}^{\Delta\tau} P(\tau)\,\mathrm{d}\tau =$$

$$= p_{\mathrm{B}}^{*} \int_{(g_t^\circ)} \left\{ \int_{(b)} \widetilde{p}_\circ^{*} \left[\int_{\tau=0}^{\Delta\tau} \dot{u}^{*a^\circ}(\tau)\, Q_{a^\circ}^{\circ * h^\circ}(\tau)\,\mathrm{d}\tau \right] \mathrm{d}x^3 \right\} \varepsilon_{\eta\vartheta 3}^{\circ} t^{\circ\vartheta} \mathrm{d}s^\circ. \tag{5.23}$$

In order to detail the integration across the thickness in (5.22) we should use equations (5.11), (5.12), (5.13)-(5.16) and (5.10).

REMARK 5.1. It is worth emphasizing that calculation of the work increment done by the tractions in an equilibrium configuration $(\overline{B})$ during a kinematically admissible displacement Δu_{B}^k is determined by the derivatives $\dot{u}_{\mathrm{B}}^k$, $\ddot{u}_{\mathrm{B}}^k$, $\dddot{u}_{\mathrm{B}}^k$,... in (5.1). Numerical computations can be performed e.g., by the finite element method.

REMARK 5.2. When the computation of ΔW is to be performed on an equilibrium path by means of the incremental form of the principle of virtual work, we have to determine $\dot{u}_{\mathrm{B}}^k$, $\ddot{u}_{\mathrm{B}}^k$, $\dddot{u}_{\mathrm{B}}^k$,... in advance as functions of $\dot{p}_{\mathrm{B}}$, $\ddot{p}_{\mathrm{B}}$, $\dddot{p}_{\mathrm{B}}$,... which define Δp_{B}. The computations can also be carried out by the finite element method.

6. **Conclusions**

The well-known Nanson formula gives a relationship between the surface elements $\mathrm{d}\mathbf{A}$ and $\mathrm{d}\overline{\mathbf{A}}$ taken at the points P and $\overline{P}$ of the reference configuration (B) and present configuration $(\overline{B})$, respectively.

Making use of the Nanson formula a relationship has been found for shells between the surface elements $\mathrm{d}\mathbf{A}^\circ$ and $\mathrm{d}\overline{\mathbf{A}}$. The surface element $\mathrm{d}\mathbf{A}^\circ$ with an arbitrary orientation is associated with the point P° of the base surface (A°) of a shell in configuration (B) while the surface element $\mathrm{d}\overline{\mathbf{A}}$ is associated with the point P of configuration $(\overline{B})$. As a special case we have investigated what happens when $\mathrm{d}\mathbf{A}^\circ$ is loacated either on the base surface (A°) or on the side surface (A^*).

The geometric variables in a small neighborhood of the equilibrium configuration $(\overline{B})$ can be given by truncated Taylor expansions with respect to a control parameter. The control parameter is regarded as quasi-time. In this way the displacement field and the surface element vectors can be regarded as functions of a control parameter (control parameters) in a small neighborhood of the configuration $(\overline{B})$. We have preferred the total Lagrangian formulation in which the variables are defined at the points of the reference configuration (B).

Using the relationship set up for the surface element vectors calculation of the work increment ΔW done by the deformation dependent normal traction $\widetilde{p}$ during a displacement increment $\Delta\overline{\mathbf{u}}$ requires calculation of integrals taken on the surface of configuration (B). The traction $\widetilde{p}$ is exerted on the surface part $(\overline{A}_t)$ of configuration $(\overline{B})$, and the displacement increment $\Delta\overline{\mathbf{u}}$ is measured in a small neighborhood of configuration $(\overline{B})$.

In special cases, i.e., for shells the surface integrals are calculated either on the base surface (A°) or on the side surface (A^*) depending on which surface part is loaded.

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