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TIME RATES OF TENSORS IN CONTINUUM MECHANICS UNDER ARBITRARY TIME DEPENDENT TRANSFORMATIONS PART II.

SYSTEMS OF MATERIALLY OBJECTIVE TIME RATES OF TENSORS

Imre Kozák
Department of Mechanics, University of Miskolc
3515 Miskolc – Egyetemváros, Hungary
mechkoz@gold.uni-miskolc.hu

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Abstract. Relations have been deduced for the material time rates of deformation tensors of a body in coordinate systems moving arbitrarily with respect to each other. The concept of materially objective time rates of tensors is associated with the concept of co-ordinates systems moving arbitrarily (the transformations are time dependent and arbitrary) which respect to each other, rather than with rigid body motion (the transformations are orthogonal) as generally accepted in the literature. It has been shown that one part of the materially objective time rates of tensors found in literature and also those proposed in this paper are materially objective for arbitrary time dependent transformations, while their other part for orthogonal transformations only.

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1. Introduction

1.1. The first part of this paper (Kozák [14]) gives the kinematic quantities for coordinate systems moving arbitrarily (capable of deformation) with respect to each other. It has also been shown how the material time rates defined in these coordinate systems are related to each other.

Making use of the results of the first part, the present second part gives a new and more general definition for the materially (physically) objective time rates of continuum mechanics. In addition, they are arranged into a system. An outline is provided about the issue how the physically objective time rates of continuum mechanics are related to each other both for those being defined in the paper and for those taken from the literature.

1.2. Section 2 is devoted to the material time rates of the tensors describing the deformations of continuum both in the fixed coordinate system and in another one moving arbitrarily with respect to the fixed one. The spin tensors of the principal directions of strains and the rotation tensor are also determined in these coordinate systems.

In Section 3 materially objective tensors determined by the velocity field and the deformation of continuum are sought and then the basic system of materially objective time rates of a tensor is defined by making use of the relations established between the material time rates defined in coordinate systems moving arbitrarily with respect to each other in Section 4 of Part I. It is also shown how the objective time rates found in the literature follow from the basic system and the objective tensors defined in the first part of the present section. In addition to this, new materially objective time rates are defined.

1.3. Notations and notational conventions are the same as in the first part. When citing equations of the first part, the equation number is followed by a point and the roman numeral "I".

2. **Material time rates of deformation tensors of a body**

2.1. For our later considerations the material time rates of the scalar line elements ds, the stretch λ_e, the deformation gradient $\boldsymbol{F}$, the rotation tensor $\boldsymbol{FR}$, the left stretch tensor $\boldsymbol{V}$ and the eigenvectors $\mathbf{n}_p$ as well as the conclusions obtained from the investigation of these rates deserve special attention.

Consider first the material time rates defined in the coordinate system $\{x^p\}$.

For the vectorial line element d$\mathbf{r}$, the scalar line element ds, the stretch λ_e and the deformation gradient $\boldsymbol{F}$ we may write the known formulae

$$ {}^{(\mathrm{x})}(\mathrm{d}\mathbf{r})^{\cdot} = \mathrm{d}^{(\mathrm{x})}\mathbf{v} = {}^{(\mathrm{x})}\boldsymbol{L}\cdot\mathrm{d}\mathbf{r}\ , \tag{2.1} $$

$$ \frac{{}^{(\mathrm{x})}(\mathrm{d}s)^{\cdot}}{\mathrm{d}s} = \mathbf{e}\cdot{}^{(\mathrm{x})}\boldsymbol{D}\cdot\mathbf{e} = e^p\ {}^{(\mathrm{x})}d_{pq}e^q\ , \tag{2.2} $$

$$ \frac{{}^{(\mathrm{x})}\dot{\lambda}_e}{\lambda_e} = {}^{(\mathrm{x})}(\ln\lambda_e)^{\cdot} = \mathbf{e}\cdot{}^{(\mathrm{x})}\boldsymbol{D}\cdot\mathbf{e}\ , \tag{2.3} $$

$$ {}^{(\mathrm{x})}\boldsymbol{F}^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{L}\cdot\boldsymbol{F}\ . \tag{2.4} $$

2.2. Let

$$ {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} = {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{W}^K{}_L\overline{\mathbf{G}}_K\overline{\mathbf{G}}^L \tag{2.5} $$

be the spin tensor referring to the coordinate system $\{\overline{\nu}^p\}$ in the reference configuration (see Subsection 3.4. of Part I.). Further let

$$ {}^{(\mathrm{nx})}\boldsymbol{W} = {}^{(\mathrm{nx})}W^p{}_q\mathbf{g}_p\mathbf{g}^q \tag{2.6} $$

be the spin tensor referring to the co-ordinate system $\{\nu^p\}$ in the present configuration. ${}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$ and ${}^{(\mathrm{nx})}\boldsymbol{W}$ are both skew tensors.

The eigenvectors $\overline{\mathbf{n}}_p$ and $\mathbf{n}_q$ of the principal axes of strains $\overline{\boldsymbol{U}}$ and $\boldsymbol{V}$ remain unit vectors during the deformation of the body and are always orthonormal to each other. Consequently, the following relations hold in the coordinate systems $\{\overline{\nu}^p\}$ and $\{\nu^p\}$ rotating together with the principal axes

$$^{(\overline{\mathrm{n}})}\dot{\overline{\mathbf{n}}}_p = \mathbf{0} \quad \text{and} \quad ^{(\mathrm{n})}\dot{\mathbf{n}}_q = \mathbf{0}\,. \tag{2.7}$$

Apply formulae (4.13.I) and determine the material time rates of the eigenvectors $\overline{\mathbf{n}}_p$ and $\mathbf{n}_q$ i.e., the corotate rates, taking into account that the motion of coordinate systems $\{\overline{\nu}^p\}$ and $\{\nu^p\}$ with respect to the coordinate system $\{x^p\}$ is a rigid body motion, i.e., $^{(\overline{\mathrm{n}}\mathrm{x})}\boldsymbol{L} = {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$ and $^{(nx)}\boldsymbol{L} = {}^{(nx)}\boldsymbol{W}$. In this way we have

$$^{(\mathrm{x})}\dot{\overline{\mathbf{n}}}_p = {}^{(\overline{\mathrm{n}})}\dot{\overline{\mathbf{n}}}_p - \overline{\mathbf{n}}_p \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} = {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \overline{\mathbf{n}}_p, \tag{2.8}$$

$$^{(\mathrm{x})}\dot{\mathbf{n}}_q = {}^{(\mathrm{n})}\dot{\mathbf{n}}_q - \mathbf{n}_q \cdot {}^{(\mathrm{nx})}\boldsymbol{W} = {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \mathbf{n}_q. \tag{2.9}$$

Following this we proceed to determine the material time rate of the expression $\mathbf{n}_p = \boldsymbol{R} \cdot \overline{\mathbf{n}}_p$ (see 3.15.I):

$$^{(\mathrm{x})}\dot{\mathbf{n}}_p = {}^{(\mathrm{x})}\boldsymbol{R}^{\cdot} \cdot \overline{\mathbf{n}}_p + \boldsymbol{R} \cdot {}^{(\mathrm{x})}\dot{\overline{\mathbf{n}}}_p.$$

After substitutions and taking into account that $\mathbf{n}_p$ can be any of the eigenvectors, we obtain:

$$^{(nx)}\boldsymbol{W} = {}^{(\mathrm{x})}\boldsymbol{R}^{\cdot} \cdot \boldsymbol{R}^{\mathrm{T}} + \boldsymbol{R} \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \boldsymbol{R}^{\mathrm{T}}. \tag{2.10}$$

Since both $^{(\mathrm{nx})}\boldsymbol{W}$ and $^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$ are skew and the product $\boldsymbol{R} \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \boldsymbol{R}^{\mathrm{T}}$ is also skew, it follows that so is $^{(\mathrm{x})}\boldsymbol{R}^{\cdot} \cdot \boldsymbol{R}^{\mathrm{T}}$. Let the corresponding spin tensor (the rate of rotation tensor) be

$$^{(\mathrm{x})}\boldsymbol{R}^{\cdot} \cdot \boldsymbol{R}^{\mathrm{T}} = {}^{(\mathrm{Rx})}\boldsymbol{W}, \tag{2.11}$$

from which we obtain the material time rate of the rotation $\boldsymbol{R}$:

$$^{(\mathrm{x})}\boldsymbol{R}^{\cdot} = {}^{(\mathrm{Rx})}\boldsymbol{W} \cdot \boldsymbol{R}. \tag{2.12}$$

As can be seen from (2.10) the spin tensors $^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}}$, $^{(\mathrm{nx})}\boldsymbol{W}$ and $^{(\mathrm{Rx})}\boldsymbol{W}$ we have introduced should meet the relation

$$^{(\mathrm{nx})}\boldsymbol{W} = {}^{(\mathrm{Rx})}\boldsymbol{W} + \boldsymbol{R} \cdot {}^{(\overline{\mathrm{n}}\mathrm{x})}\overline{\boldsymbol{W}} \cdot \boldsymbol{R}^{\mathrm{T}}. \tag{2.13}$$

2.3. According to (3.16.I) it holds for the left stretch tensor in the coordinate system $\{\nu^p\}$ rotating together with the eigenvectors that

$$^{(\mathrm{n})}\boldsymbol{V}^{\cdot} = {}^{(\mathrm{n})}\dot{\lambda}_{\underline{p}}\delta^p_q \mathbf{n}_p \mathbf{n}^q\,. \tag{2.14}$$

Therefore it follows from (4.14.I) that

$$\begin{aligned} ^{(\mathrm{x})}\boldsymbol{V}^{\cdot} &= {}^{(\mathrm{n})}\boldsymbol{V}^{\cdot} + {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \boldsymbol{V} - \boldsymbol{V} \cdot {}^{(\mathrm{nx})}\boldsymbol{W} = \\ &= {}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}}\delta^p_q \mathbf{n}_p \mathbf{n}^q + {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \boldsymbol{V} - \boldsymbol{V} \cdot {}^{(\mathrm{nx})}\boldsymbol{W}. \end{aligned} \tag{2.15}$$

Here we have utilized the relation $^{(\mathrm{n})}\dot{\lambda}_{\underline{p}} = {}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}}$ which holds in the coordinate systems $\{x^p\}$ and $\{\nu^p\}$ which perform a rigid body motion with respect to each other (For the sake of comparison it is worth recalling formula (2.21), which we present later).

2.4. Recalling the polar decomposition (3.14.I) for the deformation gradient $\boldsymbol{F}$, it follows from (2.4)

$$ {}^{(\mathrm{x})}\boldsymbol{L} = {}^{(\mathrm{x})}\boldsymbol{F}^{\cdot} \cdot \boldsymbol{F}^{-1} = \left({}^{(\mathrm{x})}\boldsymbol{V}^{\cdot} \cdot \boldsymbol{R} + \boldsymbol{V} \cdot {}^{(\mathrm{x})}\boldsymbol{R}^{\cdot} \right) \cdot \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{V}^{-1} . $$

Making use of equations (2.15) and (2.12) we have

$$ {}^{(\mathrm{x})}\boldsymbol{L} = {}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} + {}^{(\mathrm{nx})}\boldsymbol{W} + \boldsymbol{V} \left({}^{(\mathrm{Rx})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W} \right) \cdot \boldsymbol{V}^{-1} , \tag{2.16} $$

where

$$ {}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} = \left({}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}} \delta^p_q \mathbf{n}_p \mathbf{n}^q \right) \cdot \boldsymbol{V}^{-1} = {}^{(\mathrm{x})}\left(\ln_{\underline{p}} \right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q . \tag{2.17} $$

2.5. When formulating relative material time rates, we shall utilize the relations being set up between material time rates defined in the coordinate systems $\{x^p\}$ and $\{\widehat{x}^k\}$ in Section 4 of Part I.

In the case of the vectorial line element we obtain from (4.30.I), (2.1) and (3.27.I) that

$$ {}^{(\widehat{\mathrm{x}})}(\mathrm{d}\mathbf{r}^*)^{\cdot} = \mathrm{d}{}^{(\widehat{\mathrm{x}})}\mathbf{v} = {}^{(\mathrm{x})}(\mathrm{d}\mathbf{r})^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{L} \cdot \mathrm{d}\mathbf{r} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{L} \cdot \mathrm{d}\mathbf{r} . \tag{2.18} $$

For the scalar line element the result follows from the expression

$$ {}^{(\widehat{\mathrm{x}})}(\mathrm{d}s \cdot \mathrm{d}s)^{\cdot} = 2{}^{(\widehat{\mathrm{x}})}(\mathrm{d}s)^{\cdot}\,\mathrm{d}s = {}^{(\widehat{\mathrm{x}})}(\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r})^{\cdot} . $$

If we substitute equation (2.18) and take (3.28.I) also into consideration we obtain:

$$ \frac{{}^{(\widehat{\mathrm{x}})}(\mathrm{d}s)^{\cdot}}{\mathrm{d}s} = \mathbf{e} \cdot {}^{(\widehat{\mathrm{x}})}\boldsymbol{D} \cdot \mathbf{e} = \mathbf{e} \cdot \left({}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{Gx})}\boldsymbol{D} \right) \cdot \mathbf{e} = \frac{{}^{(\mathrm{x})}(\mathrm{d}s)^{\cdot}}{\mathrm{d}s} - \mathbf{e} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \mathbf{e} , \tag{2.19} $$

$$ \frac{{}^{(\widehat{\mathrm{x}})}\dot{\lambda}_e}{\lambda_e} = {}^{(\widehat{\mathrm{x}})}(\ln \lambda_e)^{\cdot} = {}^{(\mathrm{x})}(\ln \lambda_e)^{\cdot} - \mathbf{e} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \mathbf{e} . \tag{2.20} $$

Here $\mathbf{e}$ is the unit vector in the direction of the line element.

It follows from (2.17) and (2.20) that

$$ {}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} = {}^{(\mathrm{x})}\left(\ln \lambda_{\underline{p}} \right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q = \left[{}^{(\widehat{\mathrm{x}})}\left(\ln \lambda_{\underline{p}} \right)^{\cdot} + \mathbf{n}_{\underline{p}} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \mathbf{n}^{\underline{q}} \right] \delta^p_q \mathbf{n}_p \mathbf{n}^q . \tag{2.21} $$

2.6. In the case of the deformation gradient equation (4.30.I) should be applied since $\boldsymbol{F}$ has only one index relating to the present configuration. Substituting (2.4) and (3.27.I) equation

$$ {}^{(\widehat{\mathrm{x}})}(\boldsymbol{F}^*)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{F}^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \boldsymbol{F} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{L} \cdot \boldsymbol{F} \tag{2.22} $$

is obtained.

As regards the left stretch tensor $\boldsymbol{V}$, equation (4.14.I) should be followed:

$$ {}^{(\widehat{\mathrm{x}})}(\boldsymbol{V}^*{}_*)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{V}^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{D} \cdot \boldsymbol{V} + \boldsymbol{V} \cdot {}^{(\mathrm{Gx})}\boldsymbol{D} . $$

In view of equation (2.15) we find

$$\begin{aligned} {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{V}^{*}{}_{*}\right)^{\cdot} = {}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}}\delta^p_q\mathbf{n}_p\mathbf{n}^q - {}^{(\mathrm{Gx})}\boldsymbol{D}\cdot\boldsymbol{V} + \boldsymbol{V}\cdot{}^{(\mathrm{Gx})}\boldsymbol{D} + \\ + \left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right)\cdot\boldsymbol{V} - \boldsymbol{V}\cdot\left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right). \end{aligned} \tag{2.23}$$

The relative velocity gradient is obtained with the aid of equation (3.27.I). Substituting equation (2.16) we can write

$$\begin{aligned} {}^{(\widehat{\mathrm{x}})}\boldsymbol{L} = {}^{(\mathrm{x})}(\ln\boldsymbol{V})^{\circ} + \left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right) - \\ - \boldsymbol{V}\cdot\left[\left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right) - \left({}^{(\mathrm{Rx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}\right)\right] - {}^{(\mathrm{Gx})}\boldsymbol{D}, \end{aligned} \tag{2.24}$$

where

$${}^{(\mathrm{x})}(\ln\boldsymbol{V})^{\circ} = \left[{}^{(\widehat{\mathrm{x}})}\left(\ln\lambda_{\underline{p}}\right)^{\cdot} + \mathbf{n}_{\underline{p}}\cdot{}^{(\mathrm{Gx})}\boldsymbol{D}\cdot\mathbf{n}^{\underline{q}}\right]\delta^p_q\mathbf{n}_p\mathbf{n}^q .$$

Remark: The result (2.24) could also be deduced from the equation for the relative velocity gradient ${}^{(\widehat{\mathrm{x}})}\boldsymbol{L} = {}^{(\widehat{\mathrm{x}})}(\boldsymbol{F}^{*})^{\cdot}\cdot\boldsymbol{F}$, which follows from (2.22).

2.7. The well-known material time rate of the Eulerian strain tensor, i.e., the relation

$$\boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{I} - \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\cdot\boldsymbol{F}^{-1}\right] \tag{2.25}$$

valid in the coordinate system $\{x^p\}$ is obtained by utilizing equation (2.4)

$${}^{(\mathrm{x})}\boldsymbol{E}^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot{}^{(\mathrm{x})}\boldsymbol{L} . \tag{2.26}$$

Making use of equation (4.13.I) we can write in the coordinate system $\left\{\widehat{x}^k\right\}$

$${}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{E}_{**}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{D} - {}^{(\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot{}^{(\widehat{\mathrm{x}})}\boldsymbol{L} + {}^{(\mathrm{Gx})}\boldsymbol{D} . \tag{2.27}$$

2.8. Now we introduce the quantities

$${}^{(\mathrm{n}\widehat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}, \quad {}^{(\mathrm{n}\widehat{\mathrm{x}})}w^p{}_q = {}^{(\mathrm{nx})}w^p{}_q - {}^{(\mathrm{Gx})}w^p{}_q , \tag{2.28}$$

and

$${}^{(\mathrm{R}\widehat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{Rx})}\boldsymbol{W} - {}^{(\mathrm{Gx})}\boldsymbol{W}, \quad {}^{(\mathrm{R}\widehat{\mathrm{x}})}w^p{}_q = {}^{(\mathrm{Rx})}w^p{}_q - {}^{(\mathrm{Gx})}w^p{}_q . \tag{2.29}$$

where ${}^{(\mathrm{n}\widehat{\mathrm{x}})}\boldsymbol{W}$ and ${}^{(\mathrm{nx})}\boldsymbol{W}$ are both the spin tensors of the coordinate system $\{\nu^p\}$. Moreover ${}^{(\mathrm{R}\widehat{\mathrm{x}})}\boldsymbol{W}$ and ${}^{(\mathrm{Rx})}\boldsymbol{W}$ are both rates of the rotation tensor.

One can come to further conclusions from formulae (2.16) and (2.24) giving velocity gradients in the coordinate systems $\{x^p\}$ and $\left\{\widehat{x}^k\right\}$. For this purpose we shall write the formulae mentioned in the coordinate system $\{\nu^p\}$ of the principal axes by taking the additive decomposition of ${}^{(\mathrm{x})}\boldsymbol{L}$ and ${}^{(\widehat{\mathrm{x}})}\boldsymbol{L}$ also into consideration:

$${}^{(\mathrm{x})}\left(\ln\lambda_{\underline{p}}\right)^{\cdot}\delta^p_q + {}^{(\mathrm{nx})}w^p{}_q\left(1 - \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}\right) + {}^{(\mathrm{Rx})}w^p{}_q\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = {}^{(\mathrm{x})}d^p_q + {}^{(\mathrm{x})}w^p{}_q , \tag{2.30}$$

$$\begin{aligned} {}^{(\widehat{\mathrm{x}})}\left(\ln\lambda_{\underline{p}}\right)^{\cdot}\delta^p_q + {}^{(\mathrm{n}\widehat{\mathrm{x}})}w^p{}_q\left(1 - \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}\right) + {}^{(\mathrm{R}\widehat{\mathrm{x}})}w^p{}_q\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = \\ = -\left(\mathbf{n}_{\underline{p}}\cdot{}^{(\mathrm{Gx})}\boldsymbol{D}\cdot\mathbf{n}^{\underline{q}}\right)\delta^p_q + {}^{(\mathrm{x})}d^p_q + {}^{(\widehat{\mathrm{x}})}w^p{}_q . \end{aligned} \tag{2.31}$$

It should be emphasized that equations (2.30), (2.31) and formulae (2.32)-(2.39) presented later are all regarded in the coordinate system $\{\nu^p\}$ of the principal strains without drawing the reader's attention to this fact by a separate notation.

Assuming the velocity gradient ${}^{(\mathrm{x})}\boldsymbol{L}$ to be known, formulae (2.30) can be regarded as a linear system of equations with nine unknowns by which are meant the three values ${}^{(\mathrm{x})}(\ln\lambda_p)^{\cdot}$ and the three independent components of the skew tensors ${}^{(\mathrm{nx})}w^p{}_q$ and ${}^{(\mathrm{Rx})}w^p{}_q$ each. To find the unknowns from (2.30) we shall set up two sets of linear equations:

$$ {}^{(\mathrm{x})}(\ln\lambda_p)^{\cdot} = \frac{{}^{(\mathrm{x})}\dot{\lambda}_{\underline{p}}}{\lambda_{\underline{p}}} = {}^{(\mathrm{x})}d^{\underline{p}}_{p} \,, \qquad p = q \,, \tag{2.32} $$

$$ {}^{(\mathrm{nx})}w^p{}_q\left(1 - \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}\right) + {}^{(\mathrm{Rx})}w^p{}_q\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = {}^{(\mathrm{x})}d^p_q + {}^{(\mathrm{x})}w^p{}_q \,, \qquad p \neq q \,, \tag{2.33} $$

$$ -\,{}^{(\mathrm{nx})}w^p{}_q\left(1 - \frac{\lambda_{\underline{q}}}{\lambda_{\underline{p}}}\right) - {}^{(\mathrm{Rx})}w^p{}_q\frac{\lambda_{\underline{q}}}{\lambda_{\underline{p}}} = {}^{(\mathrm{x})}d^p_q - {}^{(\mathrm{x})}w^p{}_q \,, \qquad p \neq q \,. \tag{2.34} $$

Assuming different eigenvalues, i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, the solution of equation systems (2.33), (2.34) takes the form

$$ {}^{(\mathrm{nx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q - \frac{\left(\lambda_{\underline{p}}\right)^2 + \left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2}\,{}^{(\mathrm{x})}d^p_q \,, \qquad p \neq q \,, \tag{2.35} $$

$$ {}^{(\mathrm{Rx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q - \frac{\lambda_{\underline{p}} - \lambda_{\underline{q}}}{\lambda_{\underline{p}} + \lambda_{\underline{q}}}\,{}^{(\mathrm{x})}d^p_q \,, \qquad p \neq q \,. \tag{2.36} $$

The spin tensor of the coordinate system $\{\overline{\nu}^p\}$ in the reference configuration is obtained from (2.13):

$$ {}^{(\overline{\mathrm{n}}\mathrm{x})}w^K{}_L = -\,R_p{}^K\frac{2\lambda_{\underline{p}}\lambda_{\underline{q}}}{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2}\,{}^{(\mathrm{x})}d^p_q R^q{}_L, \qquad K \neq L. \tag{2.37} $$

For coinciding eigenvalues the above results change to a certain extent. If, for example, $\lambda_1 = \lambda_2 \neq \lambda_3$, then

- for $p = 1$ and $p = 2$

$$ {}^{(\mathrm{Rx})}w^1{}_2 = {}^{(\mathrm{x})}w^1{}_2 \quad \text{while} \quad {}^{(\mathrm{nx})}w^1{}_2 \quad \text{is undetermined,} $$

- for $p \neq q$ and $p, q = 2, 3$ formulae (2.35), (2.36) remain valid.

If $\lambda_1 = \lambda_2 = \lambda_3$ and $p \neq q$, then

$$ {}^{(\mathrm{Rx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q \quad \text{while} \quad {}^{(\mathrm{nx})}w^p{}_q \quad \text{is undetermined.} $$

In view of the formal similarity of (2.30) to (2.31) it follows at once by repeating the preceding line of thought that on the one hand:

$$^{(\hat{\mathrm{x}})}\left(\ln\lambda_p\right)^{\cdot} = {}^{(\hat{\mathrm{x}})}d^{\underline{p}}_{\underline{p}} = {}^{(\mathrm{x})}d^{\underline{p}}_{\underline{p}} - {}^{(\mathrm{Gx})}d^{\underline{p}}_{\underline{p}}$$

[which has already been known - see (2.20)], and on the other hand:

$$^{(\mathrm{n}\hat{\mathrm{x}})}w^p{}_q = {}^{(\hat{\mathrm{x}})}w^p{}_q - \frac{\left(\lambda_{\underline{p}}\right)^2 + \left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2}\, {}^{(\mathrm{x})}d^p_q\,, \qquad p \neq q\,, \tag{2.38}$$

$$^{(\mathrm{R}\hat{\mathrm{x}})}w^p{}_q = {}^{(\hat{\mathrm{x}})}w^p{}_q - \frac{\lambda_{\underline{p}} - \lambda_{\underline{q}}}{\lambda_{\underline{p}} + \lambda_{\underline{q}}}\, {}^{(\mathrm{x})}d^p_q\,, \qquad p \neq q\,. \tag{2.39}$$

2.9. Comparing (2.35) and (2.38) as well as (2.36) and (2.39), further results are obtained for the spin tensors:

$$^{(\hat{\mathrm{x}})}\boldsymbol{W} - {}^{(\mathrm{n}\hat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}, \tag{2.40}$$

$$^{(\hat{\mathrm{x}})}\boldsymbol{W} - {}^{(\mathrm{R}\hat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}. \tag{2.41}$$

By subtracting (2.40) from (2.41) we have

$$^{(\mathrm{n}\hat{\mathrm{x}})}\boldsymbol{W} - {}^{(\mathrm{R}\hat{\mathrm{x}})}\boldsymbol{W} = {}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}. \tag{2.42}$$

2.10. In the case when ${}^{(\mathrm{x})}d^p_q = 0$ for $p \neq q$, i.e., when the angles formed by the principal axes of the stretch tensors do not change at the instant under consideration:

$$^{(nx)}\boldsymbol{W} = {}^{(Rx)}\boldsymbol{W} = {}^{(x)}\boldsymbol{W}, \quad {}^{(\overline{n}x)}\boldsymbol{W} = 0 \quad \text{and} \quad {}^{(n\hat{x})}\boldsymbol{W} = {}^{(R\hat{x})}\boldsymbol{W} = {}^{(\hat{x})}\boldsymbol{W}. \tag{2.43}$$

3. Materially objective time rates of tensors and their system

3.1. In the present paper tensors are understood as materially (or physically) objective (or objective for short) quantities if their components follow the general transformation laws of tensors in coordinate systems moving arbitrarily (being deformable) with respect to each other.

Summarizing briefly: an objective tensor and its objective time rates are invariant under arbitrary time dependent transformations, more briefly: if they are objective, they can be defined independently of the choice of coordinate systems.

In literature invariance under orthogonal time dependent transformations is understood as the criterion for objectivity. For this reason we distinguish, in the sequel, tensors being objective under arbitrary time dependent transformation from those being objective for orthogonal time dependent transformation only.

3.2. On the basis of the results presented in Sections 2, 3 and 4 of Part I we can conclude that the tensors describing the time rate of change of tensors and being, therefore, defined in a certain co-ordinate system do not satisfy, in general, the above criterion of objectivity. As examples we could mention, among others, the velocity vector fields, the velocity gradient, the strain rate tensor, the spin tensors (3.36.I)-(3.39.I), the material time rates (4.13.I)-(4.16.I) of a tensor, and the material time rates of line elements and those of stretches (2.19), (2.20).

At the same time we can also conclude on the basis of formulae (2.40)-(2.42) that some tensors, which are defined as a difference between various spin tensors, are independent of the choice of a coordinate system even if the coordinate systems under consideration move arbitrarily with respect to each other. They are, therefore, objective.

If we restrict the criterion for objectivity to orthogonal transformations only, additional objective tensors can be obtained. In this case, ${}^{(\mathrm{Gx})}\boldsymbol{D} = 0$ and for example according to (2.19), (2.20), the material time rates of the scalar line element and the stretch are objective.

In the present Section the objective tensors are considered first and then their objective time rates are investigated.

Since the sum, difference and product of objective tensors are also objective additional objective tensors can be established with the aid of the known ones.

In what follows, marking the co-ordinate system $\{x^p\}$ as such in which the tensors are defined, we collect the objective tensors introduced in Section 2 and, in addition, we also deduce further objective tensors by means of the rule mentioned in the preceding paragraph.

The tensors and the time rates of tensors that are objective under arbitrary time dependent transformation will be separated from those being objective under orthogonal transformation only.

3.3. *Objective tensors under arbitrary time dependent transformations*:

- according to (2.40)-(2.42):

$$ {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}, \quad {}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \quad {}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \tag{3.1} $$

- according to (3.1):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{nx})}\boldsymbol{W}, \quad {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \quad \boldsymbol{V}\cdot\left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right)\cdot\boldsymbol{V}^{-1}, \tag{3.2} $$

- according to (3.2):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \quad \text{where} \quad {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} = {}^{(\mathrm{x})}\boldsymbol{D} + {}^{(\mathrm{nx})}\boldsymbol{W} - \boldsymbol{V}\cdot\left({}^{(\mathrm{nx})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right)\cdot\boldsymbol{V}^{-1}. \tag{3.3} $$

3.4. *Objective tensors under orthogonal time dependent transformations only.* In the case of an orthogonal transformation motion of the grid (i.e. of the coordinate system $\{\widehat{x}^k\}$ relative to the coordinate system $\{x^p\}$) is that of a rigid body. Consequently, in this case the tensors listed below are all objective:

- according to (3.28.I) the strain rate tensor:

$$ {}^{(\mathrm{x})}\boldsymbol{D}, \tag{3.4} $$

- according to (3.2) and (3.4):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{nx})}\boldsymbol{W}, \quad {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{Rx})}\boldsymbol{W}, \tag{3.5} $$

- according to (2.19) and (2.20):

$$ {}^{(\mathrm{x})}(\mathrm{d}s)^{\cdot}, \quad {}^{(\mathrm{x})}(\lambda_e)^{\cdot}, \quad {}^{(\mathrm{x})}(\ln\lambda_e)^{\cdot}, \tag{3.6} $$

- according to (2.17) and (3.6):

$$ {}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} = {}^{(\mathrm{x})}\left(\ln_{\underline{p}}\right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q, \tag{3.7} $$

- according to (3.2), (3.4) and (3.7)

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{II}}, \tag{3.8a} $$

where

$$ {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{II}} = {}^{(\mathrm{nx})}\boldsymbol{W} + {}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\circ} = {}^{(\mathrm{x})}\boldsymbol{V}^{\cdot} \cdot \boldsymbol{V}^{-1} + \boldsymbol{V} \cdot {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \boldsymbol{V}^{-1}, $$

and

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{III}}, \tag{3.8b} $$

where

$$ {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{III}} = {}^{(\mathrm{Rx})}\boldsymbol{W} + {}^{(\mathrm{x})}(\ln \boldsymbol{V})^{\cdot}, $$

- according to (3.5) and (3.6):

$$ {}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{IV}}, \tag{3.9} $$

where $ {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{IV}} = {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} - {}^{(\mathrm{x})}\boldsymbol{D} = {}^{(\mathrm{nx})}\boldsymbol{W} - \boldsymbol{V} \cdot \left({}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right) \cdot \boldsymbol{V}^{-1}. $

3.5. Making use of the equations (4.13.I)-(4.16.I) one can define the objective time rate of an arbitrary tensor (the term objective refers to the objectivity of the materially objective time rate as a tensor).

3.6. *Objective time rates for arbitrary time dependent transformations.* Substitution of ${}^{(Gx)}\boldsymbol{L}$ from (3.27.I) into (4.13.I) yields:

$$ {}^{(\hat{\mathrm{x}})}(\boldsymbol{A}_{**})^{\cdot} + {}^{(\hat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\hat{\mathrm{x}})}\boldsymbol{L} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}. $$

Similarly, other three relations, which are not detailed here, can be obtained from (4.14.I)-(4.16.I).

Now we can define the so called *basic system of the time rates* [written here in the coordinate system $\{x^p\}$] *being objective under any time dependent transformations* (or *the basic system of objective rates* for brevity's sake). Denoting them by a small triangle used as superscript we may write:

$$ \text{I.} \quad (\boldsymbol{A}_{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}, \tag{3.10} $$

$$ \text{II.} \quad (\boldsymbol{A}^{*}{}_{*})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}, \tag{3.11} $$

$$ \text{III.} \quad (\boldsymbol{A}_{*}{}^{*})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}, \tag{3.12} $$

$$ \text{IV.} \quad (\boldsymbol{A}^{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}. \tag{3.13} $$

We write only equation (3.11) in indicial notation:

$$\left(a^{p}{}_{q}\right)^{\nabla} = {}^{(\mathrm{x})}\left(a^{p}{}_{q}\right)^{\cdot} - {}^{(\mathrm{x})}l^{p}{}_{s}a^{s}{}_{q} + a^{p}{}_{s}\,{}^{(\mathrm{x})}l^{s}{}_{q}. \tag{3.14}$$

Equation (3.10) can be identified as the *Cotter-Rivlin rate* (1955, [8]); (3.13) as the *Oldroyd* (1950, [6]) or *Trusdell* (1955, [7]) *rate*. All the four objective rates have been given by *Atluri* (1984, [2]).

The objective time rates obey the general rules valid for addition and multiplication of tensors.

Decomposing the tensor ${}^{(\mathrm{x})}\boldsymbol{L}$ and introducing the quantity

$${}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}} = {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{W}, \tag{3.15}$$

the basic system (3.10)-(3.13) can be transformed into other form:

$$\text{I.}\quad (\boldsymbol{A}_{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}, \tag{3.16}$$

$$\text{II.}\quad (\boldsymbol{A}^{*}{}_{*})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}, \tag{3.17}$$

$$\text{III.}\quad (\boldsymbol{A}_{*}{}^{*})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}, \tag{3.18}$$

$$\text{IV.}\quad (\boldsymbol{A}^{**})^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{w}}. \tag{3.19}$$

From the basic system of objective rates additional objective rates can be deduced with the help of objective tensors (see Subsection 3.3. on page 148). Consider (3.1) as an example. Since the tensor

$$\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)$$

is also physically objective, new objective rates are obtained by adding it to the right hand side of equations (3.16)-(3.19). Following the same procedure in respect of equation (3.1), let us introduce the quantities

$${}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{wn}} = {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{W} - \left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{nx})}\boldsymbol{W}\right)$$

$${}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{wn}} = {}^{(\mathrm{nx})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{nx})}\boldsymbol{W}, \tag{3.20}$$

$${}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{wR}} = {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{W} - \left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right)\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W} - {}^{(\mathrm{Rx})}\boldsymbol{W}\right)$$

$${}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{wR}} = {}^{(\mathrm{Rx})}\boldsymbol{W}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{Rx})}\boldsymbol{W} \tag{3.21}$$

by the use of which we obtain the objective rates:

$$\text{I.}\quad (\boldsymbol{A}_{**})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}, \tag{3.22}$$

$$\text{II.}\quad (\boldsymbol{A}^{*}{}_{*})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}, \tag{3.23}$$

$$\text{III.}\quad (\boldsymbol{A}_{*}{}^{*})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}, \tag{3.24}$$

$$\text{IV.}\quad (\boldsymbol{A}^{**})^{\nabla}_{\mathrm{J}} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{A}_{\mathrm{J}}. \tag{3.25}$$

where ”J” is either ”wn” or ”wR”.

New objective rates are obtained with the help of the objective tensor (3.3). According to (3.10) and (3.3) the rate

$$\left(\boldsymbol{A}_{**}\right)_{\mathrm{I}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L} -$$

$$-\left[\left({}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}\right)^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \left({}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}\right)\right] = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}\right)^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}$$

is, for example, objective. In this way we find the following objective rates:

$$\text{I.} \quad \left(\boldsymbol{A}_{**}\right)_{\mathrm{I}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}\right)^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}, \tag{3.26}$$

$$\text{II.} \quad \left(\boldsymbol{A}^{*}{}_{*}\right)_{\mathrm{I}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}, \tag{3.27}$$

$$\text{III.} \quad \left(\boldsymbol{A}_{*}{}^{*}\right)_{\mathrm{I}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}\right)^{\mathrm{T}} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}\right)^{\mathrm{T}}, \tag{3.28}$$

$$\text{IV.} \quad \left(\boldsymbol{A}^{**}\right)_{\mathrm{I}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{I}}\right)^{\mathrm{T}}. \tag{3.29}$$

Since ${}^{(\mathrm{X})}\mathbf{v} = \mathbf{0}$ and ${}^{(\mathrm{X})}\boldsymbol{L} = \mathbf{0}$ in the convected coordinate system, making use of the convected material time rate deduced in the same manner as (4.11.I), we obtain

$$\text{I.} \quad \left(\boldsymbol{A}_{**}\right)^{\nabla} = {}^{(\mathrm{X})}\left(\boldsymbol{A}_{**}\right)^{\cdot} = \left.\frac{\partial a_{KL}}{\partial t}\right|_{(\mathrm{X})} \mathbf{G}^{K}\mathbf{G}^{L}, \tag{3.30}$$

$$\text{II.} \quad \left(\boldsymbol{A}^{*}{}_{*}\right)^{\nabla} = {}^{(\mathrm{X})}\left(\boldsymbol{A}^{*}{}_{*}\right)^{\cdot} = \left.\frac{\partial a^{K}{}_{L}}{\partial t}\right|_{(\mathrm{X})} \mathbf{G}_{K}\mathbf{G}^{L}, \tag{3.31}$$

$$\text{III.} \quad \left(\boldsymbol{A}_{*}{}^{*}\right)^{\nabla} = {}^{(\mathrm{X})}\left(\boldsymbol{A}_{*}{}^{*}\right)^{\cdot} = \left.\frac{\partial a_{K}{}^{L}}{\partial t}\right|_{(\mathrm{X})} \mathbf{G}^{K}\mathbf{G}_{L}, \tag{3.32}$$

$$\text{IV.} \quad \left(\boldsymbol{A}^{**}\right)^{\nabla} = {}^{(\mathrm{X})}\left(\boldsymbol{A}^{**}\right)^{\cdot} = \left.\frac{\partial a^{KL}}{\partial t}\right|_{(\mathrm{X})} \mathbf{G}_{K}\mathbf{G}_{L} \tag{3.33}$$

for the basic system of objective rates in convected coordinates. Equations (3.30)-(3.33) have already been given by *Sedov* (1960, [3]) and *Atluri* (1984, [2]).

3.7. *Objective time rates for orthogonal time dependent transformations.* According to (3.4) the strain rate tensor is objective for orthogonal time dependent transformations. Consequently, from equations (3.16)-(3.19) and (3.22)-(3.25) objective rates independent of the index positions are at once obtained:

$$\boldsymbol{A}_{\mathrm{w}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{W} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{x})}\boldsymbol{W}, \tag{3.34}$$

$$\boldsymbol{A}_{\mathrm{wn}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{nx})}\boldsymbol{W}, \tag{3.35}$$

$$\boldsymbol{A}_{\mathrm{wR}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{Rx})}\boldsymbol{W} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot {}^{(\mathrm{Rx})}\boldsymbol{W}. \tag{3.36}$$

Equation (3.34) can be identified as the *Jaumann rate* (1911, [1]); (3.35) as the *Soverby-Chu rate* (1984, [12]); (3.36) as the *Green-Naghdi* (1965, [9]) or *Green-McInnis* (1967, [10]) or *Dienes* (1984, [11]) or *Atluri* (1984, [2]) *rate.* These equations have also been given by *Dubey* (1987, [4]), who employed a method that he called principal axis technique.

Since ${}^{(\mathrm{x})}\boldsymbol{D}$ is objective, so is the rate

$$\boldsymbol{A}_{\mathrm{S}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot {}^{(\mathrm{x})}\boldsymbol{W} + {}^{(\mathrm{x})}D_{\mathrm{I}}\boldsymbol{A} \tag{3.37}$$

independently of the position of indices; ${}^{(\mathrm{x})}D_{\mathrm{I}}$ is the first scalar invariant of ${}^{(\mathrm{x})}\boldsymbol{D}$.

Objective rates depending on the position of indices can be obtained from the basic system (3.10)-(3.13) if we utilize tensors (3.8a)-(3.9) which are objective for orthogonal transformation only:

$$\text{I.}\quad (\boldsymbol{A}_{**})_{\mathrm{K}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}, \tag{3.38}$$

$$\text{II.}\quad ({\boldsymbol{A}^{*}}_{*})_{\mathrm{K}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}, \tag{3.39}$$

$$\text{III.}\quad ({\boldsymbol{A}_{*}}^{*})_{\mathrm{K}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + \left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}, \tag{3.40}$$

$$\text{IV.}\quad (\boldsymbol{A}^{**})_{\mathrm{K}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\left({}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}\right)^{\mathrm{T}}. \tag{3.41}$$

Here "K = II, III, IV" and the tensors ${}^{(\mathrm{x})}\boldsymbol{L}_{\mathrm{K}}$ are identified by equations (3.8a)-(3.9).

For "K = IV" equations (3.39)-(3.41) coincide with the *Balla-Szabó rate* (1988, [5]).

Using ${}^{(\mathrm{x})}D_{\mathrm{I}}$ from (3.13) we get an objective rate

$$(\boldsymbol{A}^{**})_{\mathrm{T}}^{\nabla} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}} + {}^{(\mathrm{x})}D_{\mathrm{I}}\boldsymbol{A} \tag{3.42}$$

which, when written for the stress tensor, can be identified as the *Trusdell rate* (1955, [7]).

The arithmetic mean of equations (3.42) and (3.37), i.e., the tensor

$$(\boldsymbol{A}^{**})_{\mathrm{D}}^{\nabla} = \frac{1}{2}\left((\boldsymbol{A}^{**})_{\mathrm{T}}^{\nabla} + \boldsymbol{A}_{\mathrm{S}}^{\nabla}\right) =$$

$$= {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - \left(\frac{1}{2}\,{}^{(\mathrm{x})}\boldsymbol{D} + {}^{(\mathrm{x})}\boldsymbol{W}\right)\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\left(\frac{1}{2}\,{}^{(\mathrm{x})}\boldsymbol{D} - {}^{(\mathrm{x})}\boldsymbol{W}\right) + {}^{(\mathrm{x})}D_{\mathrm{I}}\boldsymbol{A} \tag{3.43}$$

is also an objective rate which can be identified as the *Durban-Baruch rate* (1977, [13]).

3.8. In the remainder of the present Section we shall consider some particular cases.

For the metric tensor g_{pq} it follows on the basis of equations (3.10) and (3.13) that

$$(g_{pq})^{\nabla} = 2^{(\mathrm{x})}d_{pq}, \qquad (g^{pq})^{\nabla} = -\,2^{(\mathrm{x})}d^{pq} \tag{3.44}$$

which hold also for any arbitrary transformation.

For orthogonal transformation $(g_{pq})_{\mathrm{w}}^{\nabla} = (g_{pq})_{\mathrm{wn}}^{\nabla} = (g_{pq})_{\mathrm{wR}}^{\nabla} = 0.$

3.9. Assuming an orthogonal transformation, objective rates of the left stretch tensor $\boldsymbol{V}$ are provided by equations (2.15) and (3.34)-(3.36). If, in addition, we utilize equation (2.32) we get

$$\boldsymbol{V}_{\mathrm{w}}^{\nabla}=\lambda_{\underline{p}}\ {}^{(\mathrm{x})}d_{\underline{p}}^{q}\delta_{q}^{p}\mathbf{n}_{p}\mathbf{n}^{q}-\left({}^{(\mathrm{x})}\boldsymbol{W}-{}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{V}+\boldsymbol{V}\cdot\left({}^{(\mathrm{x})}\boldsymbol{W}-{}^{(\mathrm{nx})}\boldsymbol{W}\right), \qquad (3.45)$$

$$\boldsymbol{V}_{\mathrm{wn}}^{\nabla}=\lambda_{\underline{p}}\ {}^{(\mathrm{x})}d_{\underline{p}}^{q}\delta_{q}^{p}\mathbf{n}_{p}\mathbf{n}^{q}, \qquad (3.46)$$

$$\boldsymbol{V}_{\mathrm{wR}}^{\nabla}=\lambda_{\underline{p}}\ {}^{(\mathrm{x})}d_{\underline{p}}^{q}\delta_{q}^{p}\mathbf{n}_{p}\mathbf{n}^{q}-\left({}^{(\mathrm{Rx})}\boldsymbol{W}-{}^{(\mathrm{nx})}\boldsymbol{W}\right)\cdot\boldsymbol{V}+\boldsymbol{V}\cdot\left({}^{(\mathrm{Rx})}\boldsymbol{W}-{}^{(\mathrm{nx})}\boldsymbol{W}\right). \qquad (3.47)$$

Making use of relations (2.35) and (2.36) we write formulae (3.45) and (3.47) entirely in the coordinate system $\{\nu^p\}$ of the principal axes:

$$\boldsymbol{V}_{\mathrm{w}}^{\nabla}=\lambda_{\underline{p}}\ {}^{(\mathrm{x})}d_{\underline{p}}^{q}\delta_{q}^{p}\mathbf{n}_{p}\mathbf{n}^{q}+\frac{\left(\lambda_{\underline{p}}\right)^{2}+\left(\lambda_{\underline{q}}\right)^{2}}{\lambda_{\underline{p}}+\lambda_{\underline{q}}}\left({}^{(\mathrm{x})}d_{q}^{p}-{}^{(\mathrm{x})}d_{\underline{p}}^{q}\delta_{q}^{p}\right)\mathbf{n}_{p}\mathbf{n}^{q}, \qquad (3.48)$$

$$\boldsymbol{V}_{\mathrm{wR}}^{\nabla}=\lambda_{\underline{p}}\ {}^{(\mathrm{x})}d_{\underline{p}}^{q}\delta_{q}^{p}\mathbf{n}_{p}\mathbf{n}^{q}+\frac{2\lambda_{\underline{p}}\lambda_{\underline{q}}}{\lambda_{\underline{p}}+\lambda_{\underline{q}}}\left({}^{(\mathrm{x})}d_{q}^{p}-{}^{(\mathrm{x})}d_{\underline{p}}^{q}\delta_{q}^{p}\right)\mathbf{n}_{p}\mathbf{n}^{q}. \qquad (3.49)$$

3.10. As regards the Eulerian strain tensor, from equations (3.10) and (2.26) we get for arbitrary time dependent transformations that

$$\left(\boldsymbol{E}_{**}\right)^{\nabla}={}^{(\mathrm{x})}\boldsymbol{D}\,. \qquad (3.50)$$

3.11. For orthogonal time dependent transformations the Jaumann rate is obtained from equations (3.34) and (2.26):

$$\begin{aligned}\boldsymbol{E}_{\mathrm{w}}^{\nabla}&={}^{(\mathrm{x})}\boldsymbol{E}^{\cdot}-{}^{(\mathrm{x})}\boldsymbol{W}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot{}^{(\mathrm{x})}\boldsymbol{W}=\\&={}^{(\mathrm{x})}\boldsymbol{D}-{}^{(\mathrm{x})}\boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot{}^{(\mathrm{x})}\boldsymbol{D}.\end{aligned} \qquad (3.51)$$

Let us generalize formulae (3.34)-(3.36) for the Eulerian Hill's strain tensor:

$$\widetilde{\boldsymbol{E}}=f\left(\lambda_{\underline{p}}\right)\delta_{q}^{p}\mathbf{n}_{p}\mathbf{n}^{q}\,, \qquad (3.52)$$

and for arbitrary spin tensors ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ (${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ is a skew tensor):

$$\widetilde{\boldsymbol{E}}^{\nabla}=\widetilde{\boldsymbol{E}}^{\cdot}+\widetilde{\boldsymbol{E}}\cdot{}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}-{}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}\cdot\widetilde{\boldsymbol{E}}. \qquad (3.53)$$

One can raise the following question: what tensors $\widetilde{\boldsymbol{E}}$ and ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ satisfy equation

$$\widetilde{\boldsymbol{E}}^{\nabla}=\widetilde{\boldsymbol{E}}^{\cdot}+\widetilde{\boldsymbol{E}}\cdot{}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}-{}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}\cdot\widetilde{\boldsymbol{E}}={}^{(\mathrm{x})}\boldsymbol{D} \qquad (3.54)$$

which is formally objective.

As Reinhardt and Dubey [16] have shown if

$$\widetilde{\boldsymbol{E}}=\ln\boldsymbol{V}=\ln\lambda_{\underline{p}}\delta_{q}^{p}\mathbf{n}_{p}\mathbf{n}^{q}\,,$$

then

$${}^{(\mathrm{x})}\widetilde{w}^{p}{}_{q}={}^{(\mathrm{nx})}w^{p}{}_{q}+\frac{1}{\ln\lambda_{\underline{p}}-\ln\lambda_{\underline{q}}}\ {}^{(\mathrm{x})}d_{q}^{p},\quad \lambda_{\underline{p}}\neq\lambda_{\underline{q}},\quad p\neq q, \qquad (3.55)$$

in the coordinate system $\{\nu^p\}$, where according to equation (2.35)

$$ {}^{(\mathrm{nx})}w^p{}_q = {}^{(\mathrm{x})}w^p{}_q - \frac{\left(\lambda_{\underline{p}}\right)^2 + \left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2} {}^{(\mathrm{x})}d^p_q, \quad \lambda_{\underline{p}} \neq \lambda_{\underline{q}}, \quad p \neq q\,, $$

and ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ is the so-called logarithmic spin tensor. The second term on the right side can be manipulated further:

$$ \frac{\left(\lambda_{\underline{p}}\right)^2 - \left(\lambda_{\underline{q}}\right)^2}{\left(\lambda_{\underline{p}}\right)^2 + \left(\lambda_{\underline{q}}\right)^2} = \frac{\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} - \frac{\lambda_{\underline{q}}}{\lambda_{\underline{p}}}}{\frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} + \frac{\lambda_{\underline{q}}}{\lambda_{\underline{p}}}} = \frac{e^{\ln \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}} - e^{-\ln \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}}}{e^{\ln \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}} + e^{-\ln \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}}}} = \tanh \ln \frac{\lambda_{\underline{p}}}{\lambda_{\underline{q}}} = \tanh\left(\ln \lambda_{\underline{p}} - \ln \lambda_{\underline{q}}\right)\,. $$

If we take the foregoing into account, equation (3.55) assumes the form:

$$ {}^{(\mathrm{x})}\widetilde{w}^p{}_q = {}^{(\mathrm{x})}w^p{}_q + \left(\frac{1}{\ln \lambda_{\underline{p}} - \ln \lambda_{\underline{q}}} - \frac{1}{\tanh\left(\ln \lambda_{\underline{p}} - \ln \lambda_{\underline{q}}\right)} \right) {}^{(\mathrm{x})}d^p_q, \quad \lambda_{\underline{p}} \neq \lambda_{\underline{q}}, \quad p \neq q. \tag{3.56} $$

Xiao at all. [15] have shown that equation (3.54) has a solution if and only if $\widetilde{\boldsymbol{E}} = \ln \boldsymbol{V}$. Summarizing what has been said above, we can conclude that equation (3.54) has the following solution

$$ (\ln \boldsymbol{V})^{\nabla} = (\ln \boldsymbol{V})^{\cdot} + \ln \boldsymbol{V} \cdot {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}} - {}^{(\mathrm{x})}\widetilde{\boldsymbol{W}} \cdot \ln \boldsymbol{V} = {}^{(\mathrm{x})}\boldsymbol{D}, \tag{3.57} $$

where ${}^{(\mathrm{x})}\widetilde{\boldsymbol{W}}$ is given by equation (3.56) and with regard to (2.9)

$$ \begin{aligned} (\ln \boldsymbol{V})^{\cdot} &= \left(\ln \lambda_{\underline{p}}\right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q + \ln \lambda_{\underline{p}} \left(\dot{\mathbf{n}}_p \mathbf{n}^q + \mathbf{n}_p \dot{\mathbf{n}}^q\right) = \\ &= \left(\ln \lambda_{\underline{p}}\right)^{\cdot} \delta^p_q \mathbf{n}_p \mathbf{n}^q + {}^{(\mathrm{nx})}\boldsymbol{W} \cdot \ln \boldsymbol{V} - \ln \boldsymbol{V} \cdot {}^{(\mathrm{nx})}\boldsymbol{W}. \end{aligned} \tag{3.58} $$

The objective time rate tensor $(\ln \boldsymbol{V})^{\nabla}$ is the so-called logarithmic strain rate tensor which can be obtained by comparing formulae (3.55), (3.57) and (3.58). It is remarkable that $(\ln \boldsymbol{V})^{\cdot} \neq (\ln \boldsymbol{V})^{\circ}$.

4. **Concluding remarks**

In the present paper invariance under arbitrary time dependent transformations (valid for co-ordinate systems moving arbitrarily with respect to each other) is regarded as a criterion for material objectivity of tensors. This means that the components of materially objective tensors and those of their objective time rates follow the general transformation rules valid for tensors in coordinate systems moving arbitrarily with respect to each other. In these cases the matrix of transformation is a function of time. The investigations that have been carried out are based on the relations we established in Part I (Kozák [14]) between material time rates defined in various co-ordinate systems.

Section 2 of the present Part II treats the material time rates of tensors describing the deformation of the continuum both in the fixed co-ordinate system and in the co-ordinate system moving arbitrarily with respect to the fixed one. The spin tensors of the principal directions of strains and of the rotation tensor are also determined in both co-ordinate systems. In this way we introduce some objective tensors (being objective partly for arbitrary partly for orthogonal time dependent transformations) each of which is determined directly or indirectly by the left stretch tensor and the velocity gradient.

Section 3 is devoted to materially objective time rates of tensors. By applying the relations, which have been deduced for the material time rates taken in coordinate systems moving arbitrarily with respect to each other, to an arbitrary tensor we have found a basic system of the objective time rates which is valid for arbitrary time dependent transformations and can be given in any coordinate system. Regarding the basic system as a point of departure and making use of the objective tensors which describe the state of velocity and deformation of a continuum, we have established objective time rates being partly new and partly published in literature.

It has been shown that one part of the objective time rates found in literature is objective under arbitrary time dependent transformation, while the other part under orthogonal transformation only.

Although the chain of thought is detailed for second-order tensors only, the non-particular results are valid for a tensor of any order.

The paper restricts its attention to the issue of objectivity of time rates and disregards the part these rates play in constitutive equations.

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