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A MAPPING TECHNIQUE FOR A HEAT CONDUCTION PROBLEM ON MOVING MESH USING THE *HP*-VERSION OF THE FINITE ELEMENT METHOD

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Abstract. The numerical solution of the thermal part of a coupled thermo-mechanical contact problem is presented. The *hp*-version finite element method is used for the discretization of the temperature field in space, and finite difference method in time. The mesh is modified during the computation, therefore the temperature field has to be transferred between different meshes. A mapping technique is developed for the *p*-version of the finite element method. Numerical examples are presented.

Mathematical Subject Classification: 74S05
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1. Introduction

This paper is concerned with the numerical solution of the thermal part of the coupled thermo-mechanical contact problem. Many papers deal with contact problems but without thermo-dynamical effects [1, 2, 3]. When friction is considered and the bodies slide on each other, heat generation and heat conduction have to be taken into account.

The contact problems are usually treated with the *h*-version of the finite element method. In recent years some papers tried to treat the contact problem with the *p*-version of the finite element method [1, 2, 4, 5]. The advantages of the *p*-version are the higher accuracy, faster convergence and coarser mesh [6]. The numerical solution of the heat conduction problem, (both the *h*- and *p*-versions), is well known [6, 7], but the coupling with the mechanical contact problem causes some difficulties. During the computation of the mechanical problem, a minor modification of the mesh is needed to avoid oscillations in the numerical solution [1, 2]. Heat conduction is a temporal process, and the temporal part of the discretization is usually treated with the finite difference method. The temperature field of the previous time step is needed

to determine the current temperature field. The temperature field of the previous and current time steps appears in the same system of equations, that is why they have to be given in the same nodal points, i.e. in the same mesh. When the mesh is modified, the temperature parameters have to be transferred from the old mesh to the new mesh. There are many mapping techniques in the literature, but mostly for the h-version. Some papers dealing with plastic deformation apply mapping techniques for data transfer between different meshes [8, 9, 10, 11]. In our investigation a mapping technique is developed for the p-version.

In this paper three dimensional axisymmetrical bodies are investigated with a two dimensional mathematical model. Numerical examples are presented.

2. The heat conduction problem

Let us consider two disjoint regions $\Omega^e\,(e=1,2)$ with piecewise smooth boundaries $\partial\Omega^e\,(e=1,2)$ occupied by two continuous deformable bodies. We intend to investigate the heat conduction during the sliding contact of the bodies. Let the initial temperature be equal to the ambient temperature everywhere in Ω^e. The surfaces of the bodies can be split into two parts: the contact area of the bodies, which is denoted by $\Gamma_c^e\,(\Gamma_c^e \in \partial\Omega^e)$, and the rest of the surfaces, which is denoted by $\Gamma_s^e\,(\Gamma_s^e \in \partial\Omega^e)$, where $\Gamma_c^e \cup \Gamma_s^e = \partial\Omega^e$ and $\Gamma_c^e \cap \Gamma_s^e = \emptyset$. When the bodies slide on each other, heat is generated on the surfaces Γ_c^e due to friction. The generated heat partially flows into the bodies and partially flows across the surfaces Γ_s^e. During this heat conduction process the temperature distribution of the bodies changes. To determine the temperature distribution $T^e(\mathbf{r},t)$ the equation of heat conduction can be applied

$$\rho^e c^e \dot{T}^e(\mathbf{r},t) = \nabla\cdot(k^e \nabla T^e(\mathbf{r},t)) + Q^e \qquad \mathbf{r}\in\Omega^e \tag{2.1}$$

where $\mathbf{r}$ is the position vector, t is time, ρ^e is density, c^e is the specific heat, k^e is thermal conductivity, Q^e is the generated heat within Ω^e and $e=1,2$. The temperature distribution $T^e(\mathbf{r},t)$ has to satisfy the following initial and boundary conditions:

$$T^e(\mathbf{r},0) = T_0^e(\mathbf{r}) \qquad \mathbf{r}\in\Omega^e \tag{2.2}$$

$$q_c^e = -k\nabla T^e(\mathbf{r},t)\cdot\mathbf{n}^e = q_{fr}^e(\mathbf{r},t) + q_{ex}^e(\mathbf{r},t) \quad \mathbf{r}\in\Gamma_c^e \tag{2.3}$$

$$q_s^e = -k\nabla T^e(\mathbf{r},t)\cdot\mathbf{n}^e = q_{co}^e(\mathbf{r},t) \qquad \mathbf{r}\in\Gamma_s^e \tag{2.4}$$

where $T_0^e(\mathbf{r})$ is the initial temperature, $\mathbf{n}^e$ is the outward normal unit vector of $\partial\Omega^e$, q_c^e is the heat flux flowing out of the e^{th} body through Γ_c^e and q_s^e is the heat flux flowing out of the e^{th} body through Γ_s^e. The heat flux q_c^e consists of two parts. The first part arises from the frictional dissipation:

$$q_{fr}^1(\mathbf{r},t) = -c_D\beta v(\mathbf{r},t)p(\mathbf{r},t) \quad \text{and} \quad q_{fr}^2(\mathbf{r},t) = -c_D(1-\beta)v(\mathbf{r},t)p(\mathbf{r},t), \tag{2.5}$$

where c_D a constant from experiment ($c_D \le 1$), $\beta = \dfrac{k^1}{k^1+k^2}$ and $0\le\beta\le 1$, $v(\mathbf{r},t)$ and $p(\mathbf{r},t)$ are the relative velocity of the bodies and the contact pressure at the point

$\mathbf{r}$ and at time t, respectively. The second part is the heat exchange between the bodies arising from the different temperatures of the bodies at the contact interface:

$$q^e_{ex}(\mathbf{r},t) = \hat{\alpha}(T^e(\mathbf{r},t) - T^{e^*}(\mathbf{r},t)), \tag{2.6}$$

where $\hat{\alpha}$ is the coefficient of surface heat transfer between the bodies, e and e^* are the number of the bodies. If $e=1$, then $e^*=2$, and if $e=2$, then $e^*=1$. The heat flux q^e_{co} denotes convective heat flux, defined as

$$q^e_{co}(\mathbf{r},t) = \alpha^e(T^e(\mathbf{r},t) - T_\infty), \tag{2.7}$$

where α^e is a coefficient of surface heat transfer and T_∞ is the specified ambient temperature of the surrounding medium. The area of the contact surface Γ^e_c and the contact pressure can be obtained by solving the contact problem with the appropriate boundary conditions.

3. The weak formulation of the heat conduction problem

The weak formulation of the above mentioned heat conduction problem can be obtained by applying Galerkin's method. Let us multiply equation (2.1) by virtual temperature Θ^e and integrate it on Ω^e:

$$\sum_{e=1}^{2} \int_{\Omega^e} (\rho^e c^e \dot{T}^e - \nabla\cdot(k^e \nabla T^e) - Q^e)\Theta^e d\Omega = 0\,.$$

After integrating by parts and employing the Gauss theorem, the weak form will be the following:

$$\sum_{e=1}^{2} \left[\int_{\Omega^e} \rho^e c^e \dot{T}^e \Theta^e d\Omega + \int_{\partial\Omega^e} q^e_n \Theta^e d\Gamma + \int_{\Omega^e} k^e \nabla T^e \cdot \nabla\Theta^e d\Omega - \int_{\Omega^e} Q^e \Theta^e d\Omega \right] \tag{3.1}$$

where $q^e_n = -k^e \nabla T \cdot \mathbf{n}^e$ is the heat flux orthogonal to the surfaces of the bodies, and positive if the heat flows out of the bodies. The bodies can exchange heat with each other through the surfaces Γ^e_c. To take into account this effect and the other boundary conditions, equation (3.1) must be specialized to bodies 1 and 2. The superscript indices denote the appropriate bodies:

$$\begin{aligned}
&\int_{\Omega^1} \rho^1 c^1 \dot{T}^1 \Theta^1 d\Omega + \int_{\Omega^1} k^1 \nabla T^1 \cdot \nabla\Theta^1 d\Omega + \int_{\Gamma^1_s} \alpha^e (T^1 - T_\infty)\Theta^1 d\Gamma - \int_{\Gamma^1_c} c_D \beta v p \Theta^1 d\Gamma + \\
&\quad + \int_{\Gamma^1_c} \hat{\alpha}(T^1 - T^2)\Theta^1 d\Gamma + \int_{\Omega^2} \rho^2 c^2 \dot{T}^2 \Theta^2 d\Omega + \int_{\Omega^2} k^2 (\nabla T^2 \cdot \nabla\Theta^2) d\Omega + \\
&\quad + \int_{\Gamma^2_s} \alpha^e (T^2 - T_\infty)\Theta^2 d\Gamma - \int_{\Gamma^2_c} c_D (1-\beta) v p \Theta^2 d\Gamma + \int_{\Gamma^2_c} \hat{\alpha}(T^2 - T^1)\Theta^2 d\Gamma = 0\,.
\end{aligned} \tag{3.2}$$

4. **Finite element discretization**

The temperature field T^e $(e = 1, 2)$ is a function of the position vector and the time. We cannot give the exact solution of (3.2) in a closed form, that is why T^e can be determined only approximately. To find an approximation to T^e, we create a set of functions by subdividing Ω^e into a number of domains, called finite elements. We define a set of basis functions on Ω in such a way that each of the basis functions is nonzero over individual elements. Applying orthogonal basis functions the round off error can be minimized [6]. In this way the temperature distribution can be written as:

$$T^e(\mathbf{r}, t) = \sum_{i=1}^{n^e} \sum_{j=1}^{N} T_{ij}^e(t) N_j(\mathbf{r}) \,, \tag{4.1}$$

where n^e is the number of finite elements of the body e, N is the number of shape functions, $N_j(\mathbf{r})$ are the shape functions and T_{ij}^e are the nodal temperatures and parameters of the i^{th} element. In a similar way, the virtual temperature also can be approximated:

$$\Theta^e(\mathbf{r}, t) = \sum_{i=1}^{n^e} \sum_{j=1}^{N} \Theta_{ij}^e(t) N_j(\mathbf{r}) \tag{4.2}$$

We restrict our investigation to axially symmetric problems, where a three dimensional problem can be treated with a two dimensional mathematical model. Using the summation convention

$$\sum_{j=1}^{N} T_{ij}^e N_j(\mathbf{r}) \equiv T_{ij}^e N_j(\mathbf{r}),$$

the functional (3.2) can be written with cylindrical coordinates and with the approximated temperature $T^e(\mathbf{r}, t)$ and virtual temperature $\Theta^e(\mathbf{r}, t)$ in the following form:

$$\begin{aligned}
0 = 2\pi \sum_{q=1}^{n^1} \Bigg[& \int_{\Omega^1} \rho^1 c^1 N_i \dot{T}_{qi}^1 N_k \Theta_{qk}^1 r dr dz + \int_{\Omega^1} k^1 \left(\frac{\partial N_i}{\partial r} T_{qi}^1 \frac{\partial N_k}{\partial r} \Theta_{qk}^1 + \frac{\partial N_i}{\partial z} T_{qi}^1 \frac{\partial N_k}{\partial z} \Theta_{qk}^1 \right) r dr dz + \\
& + \int_{\Gamma_s^1} \alpha^1 (N_i T_{qi}^1 - T_\infty) N_k \Theta_{qk}^1 r ds - \int_{\Gamma_c^1} c_D \beta v p N_i \Theta_{qi}^1 r ds + \int_{\Gamma_c^1} \hat{\alpha} (N_i T_{qi}^1 - N_k T_{q^* k}^2) N_l \Theta_{ql}^1 r ds \Bigg] + \\
+ 2\pi \sum_{q=1}^{n^2} \Bigg[& \int_{\Omega^2} \rho^2 c^2 N_i \dot{T}_{qi}^2 N_k \Theta_{qk}^2 r dr dz + \int_{\Omega^2} k^2 \left(\frac{\partial N_i}{\partial r} T_{qi}^2 \frac{\partial N_k}{\partial r} \Theta_{qk}^2 + \frac{\partial N_i}{\partial z} T_{qi}^2 \frac{\partial N_k}{\partial z} \Theta_{qk}^2 \right) r dr dz + \\
& + \int_{\Gamma_s^2} \alpha^2 (N_i T_{qi}^2 - T_\infty) N_k \Theta_{qk}^2 r ds - \int_{\Gamma_c^2} c_D (1-\beta) v p N_i \Theta_{qi}^2 r ds - \int_{\Gamma_c^2} \hat{\alpha} (N_i T_{qi}^2 - N_k T_{q^* k}^1) N_l \Theta_{ql}^2 r ds \Bigg]
\end{aligned} \tag{4.3}$$

where $ds = \sqrt{dr^2 + dz^2}$. The finite elements at the contact zone are in front of each other, so that the nodes of this elements are in coincidence. If the q^{th} element is in

the contact zone, q^* concerns the element which is in the other body in front of the q^{th} element. Let us use the following notation:

$$M_{ik}^{(1)} = \rho^1 c^1 \int\limits_{\Omega^1} N_i N_k r dr dz \qquad M_{ik}^{(2)} = \rho^2 c^2 \int\limits_{\Omega^2} N_i N_k r dr dz$$

$$\tilde{K}_{ik}^{(1)} = k^1 \int\limits_{\Omega^1} \left(\frac{\partial N_i}{\partial r}\frac{\partial N_k}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_k}{\partial z} \right) r dr dz \qquad \tilde{K}_{ik}^{(2)} = k^2 \int\limits_{\Omega^2} \left(\frac{\partial N_i}{\partial r}\frac{\partial N_k}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_k}{\partial z} \right) r dr dz$$

$$\tilde{C}_{ik}^{(1)} = \alpha^1 \int\limits_{\Gamma_s^1} N_i N_k r ds \qquad \tilde{C}_{ik}^{(2)} = \alpha^2 \int\limits_{\Gamma_s^2} N_i N_k r ds$$

$$\tilde{\tilde{C}}_{ik}^{(1)} = \hat{\alpha} \int\limits_{\Gamma_c^1} N_i N_k r ds \qquad \tilde{\tilde{C}}_{ik}^{(2)} = \hat{\alpha} \int\limits_{\Gamma_c^2} N_i N_k r ds$$

$$\tilde{F}_i^{(1)} = c_D \beta \int\limits_{\Gamma_c^1} v p N_i r ds \qquad \tilde{F}_i^{(2)} = c_D (1-\beta) \int\limits_{\Gamma_c^2} v p N_i r ds$$

For the sake of simplicity let T_∞ be equal to zero. Obviously, for the element whose side is not on Γ_c^e the integral in $\tilde{\tilde{C}}_{ik}^{(e)}$ and $\tilde{F}_i^{(e)}$ vanish. With the help of this notation and the fact that equation (4.3) can be split into two parts associated with coefficients Θ_{ik}^1, Θ_{ik}^2 we have

$$\begin{aligned} &\sum_{q=1}^{n^1} \left(M_{ik}^{(1)} \dot{T}_{qk}^1 + \tilde{K}_{ik}^{(1)} T_{qk}^1 + \tilde{C}_{ik}^{(1)} T_{qk}^1 + \tilde{\tilde{C}}_{ik}^{(1)} T_{qk}^1 - \tilde{\tilde{C}}_{ik}^{(1)} T_{q^*k}^2 - \tilde{F}_i^{(1)} \right) = 0 \\ &\sum_{q=1}^{n^2} \left(M_{ik}^{(2)} \dot{T}_{qk}^2 + \tilde{K}_{ik}^{(2)} T_{qk}^2 + \tilde{C}_{ik}^{(2)} T_{qk}^2 + \tilde{\tilde{C}}_{ik}^{(2)} T_{qk}^2 - \tilde{\tilde{C}}_{ik}^{(2)} T_{q^*k}^1 - \tilde{F}_i^{(2)} \right) = 0 \end{aligned} \tag{4.4}$$

We have to distinguish two cases. In the first one the elements are not situated on Γ_c^e. In this case the integrals $\tilde{\tilde{C}}_{ik}^{(e)}$ and $\tilde{F}_i^{(e)}$ vanish, and equation (4.4) can be written in a simpler form.

$$\sum_{q=1}^{n_n^e} (\underbrace{M_{ik}^{(e)}}_{{}^{(q)}M_{ik}} \underbrace{\dot{T}_{qk}^e}_{\dot{T}_{qk}} + \underbrace{(\tilde{K}_{ik}^{(e)} + \tilde{C}_{ik}^{(e)})}_{{}^{(q)}\tilde{K}_{ik}} \underbrace{T_{qk}^e}_{T_{qk}}) = 0 \tag{4.5}$$

where n_n^e is the number of the elements, which are not situated in the contact zone. In the second case the elements situated on Γ_c^e are considered. Now the integrals in $\tilde{\tilde{C}}_{ik}^{(e)}$ and $\tilde{F}_i^{(e)}$ do not vanish. Because of the coupling in equation (4.4) two systems of linear equations have to be computed simultaneously. These equations concern the elements situated in front of each other in Γ_c^e.

$$\sum_{q=1}^{n_c^e} \left\{ \underbrace{\begin{bmatrix} M_{ik}^{(1)} & 0 \\ 0 & M_{ik}^{(2)} \end{bmatrix}}_{{}^{(q)}M_{ik}} \underbrace{\begin{bmatrix} \dot{T}_{qk}^1 \\ \dot{T}_{q^*k}^2 \end{bmatrix}}_{\dot{T}_{qk}} + \underbrace{\begin{bmatrix} \tilde{K}_{ik}^{(1)} + \tilde{C}_{ik}^{(1)} + \tilde{\tilde{C}}_{ik}^{(1)} & -\tilde{\tilde{C}}_{ik}^{(1)} \\ -\tilde{\tilde{C}}_{ik}^{(2)} & \tilde{K}_{ik}^{(2)} + \tilde{C}_{ik}^{(2)} + \tilde{\tilde{C}}_{ik}^{(2)} \end{bmatrix}}_{{}^{(q)}\tilde{K}_{ik}} \underbrace{\begin{bmatrix} T_{qk}^1 \\ T_{q^*k}^2 \end{bmatrix}}_{T_{qk}} - \underbrace{\begin{bmatrix} \tilde{F}_i^{(1)} \\ \tilde{F}_i^{(2)} \end{bmatrix}}_{{}^{(q)}\tilde{F}_i} \right\} = 0 \tag{4.6}$$

where q runs from 1 to the number of contacting elements of one body (e.g. n_c^1), and q^* means the element in front of the actual element (q), and it is situated in the other body. The superscript (q) on the left concerns the numbering of the elements. ${}^{(q)}M_{ik}$, ${}^{(q)}\tilde{K}_{ik}$ and ${}^{(q)}\tilde{F}_i$ must be computed for every element and be added to the global convection matrix M_{ik}, conductivity matrix $\tilde{K}_{ik}$ and load vector $\tilde{F}_i$, respectively. Obviously, the element level temperature parameters are added to a global vector of temperature parameters (T_k). Equations (4.5) and (4.6) are first order differential equations in time. The usual way of discretizing such equations is the so-called ϑ-*method*

$$M_{ik}\frac{T_k^{\{n+1\}}-T_k^{\{n\}}}{\Delta t}+\tilde{K}_{ik}(\vartheta T_k^{\{n+1\}}+(1-\vartheta)T_k^{\{n\}})-(\vartheta\tilde{F}_i^{\{n+1\}}+(1-\vartheta)\tilde{F}_i^{\{n\}})=0$$

where the superscript indices in brace denote the number of time steps, Δt means the length of the time step and ϑ is a real number between 0 and 1. Since $T_k^{\{n\}}$ is known from the previous time step, a system of linear equations is obtained:

$$\underbrace{(M_{ik}+\Delta t\tilde{K}_{ik}\vartheta)}_{K_{ik}}\underbrace{T_k^{\{n+1\}}}_{T_k}-\underbrace{((M_{ik}-\Delta t\tilde{K}_{ik}(1-\vartheta))T_k^{\{n\}}+\Delta t(\vartheta\tilde{F}_i^{\{n+1\}}+(1-\vartheta)\tilde{F}_i^{\{n\}}))}_{F_i}=0\,, \tag{4.7}$$

The final system of linear equations has the form

$$K_{ik}T_k=F_i \tag{4.8}$$

where the indices i and k run from 1 to m, i.e., the number of degrees of freedom of the problem.

5. **A mapping technique**

Let us consider an axisymmetrical contact problem. The two contacting bodies slide over each other, which causes heat generation and wear on the contact surface. The generated heat flows into the bodies, therefore the temperature distribution of the bodies changes. The heat conduction equation has to be solved because the temperature field is needed for computing the thermal expansion, which is part of the displacement field. Due to the wear and the thermal expansion, the contact area changes continuously. When the contact problem is solved, the boundary of the contact zone is unknown *a priori*. In order to treat such contact problems, an adaptive *hp*-version method is required [1, 2].

One of the advantages of the p-version is that for smooth problems only coarse meshes are needed, since the error in energy norm decreases exponentially when the polynomial degree of elements is increased [6]. When the p-version is used, then the accuracy is typically high enough for the singularities to induce oscillations in the numerical solution. In the boundary of the contact region the normal stress may have jumps in its derivatives, which are regarded as singularities. Concerning the finite element discretization, we have a problem of category C [6], i.e., the mesh in 2D cannot be constructed so that the points, where the solutions are not analytic, are at nodal points. Our aim is to convert the problem of category C into one of category B

[6]. The oscillations are minimized when nodes (or nodes and edges) in two dimensions (three dimensional problems) are located at the boundary of the contact zone [1].

The coupled thermo-mechanical contact problem can be solved via the operator split technique [12]. The problem is split into separately solved thermal and mechanical parts. A possible way to solve the mechanical part is detailed in [1, 12]. In this paper we focus our attention on the thermal part.

The heat conduction problem is both a spatial and a temporal process, which are discretized by the finite element method and the finite difference method, respectively [7]. Computation of a time step consists of the computation of the contact problem, and after that the computation of the heat conduction problem, iteratively. The heat conduction problem has to be solved in the modified, new mesh, where the nodal points are located on the new boundary of the contact region. For the determination of the new temperature field, the temperature field given in the old mesh is needed. There are several mapping techniques in the literature (e.g. in connection with plastic deformation), but mostly for the h-version. For a high order approximation in FEM one can find two approaches in the literature. The first is based on the Lagrangian polynomials, the shape functions are associated with nodal point parameters. The nodal point parameter is the actual value of the field in question at the location of the node. (We can speak of e.g. nodal displacement, nodal temperature, and so on.) For the rest of the nodes with the same shape function the field value is zero. The state variables (e.g. temperature) can be easily interpolated from the nodal points of the old mesh to the nodal points of the new mesh [8, 9, 10, 11].

In the second approach, which is applied here, Legendre polynomials are used for the approximation. It should be emphasized that the shape functions constructed by the Legendre polynomials are not associated with nodal parameters. We have so called side modes, internal modes (bubble modes), edge modes and so on. That is, the parameters of the shape functions do not correspond to a field value in a specific location. Therefore the parameters have no direct physical meaning contrary to the nodal parameters. We can speak only of parameters associated with displacements and temperatures [6]. The least-squares method is more suitable for determining these parameters. The least-squares method has already been applied successfully for the h-version, where the mapped fields have to respect the local and the global equilibriums [13], or when the mesh is coarsed [14]. The data transfer can be applied similarly in the p-version via the least-squares method as has been done in the above mentioned h-version.

When the mesh modification is performed, the new temperature field can be solved by equation (4.7). In this equation F_i depends on the temperature field of the previous time step, which is given in the old mesh, hence the temperature field has to be transferred to the new mesh. The nodal values and parameters of the temperature field have to be known in the new mesh to perform the matrix multiplication in F_i.

These values can be determined by the least-squares method:

$$\sum_{i=1}^{n} \iint\limits_{S^i} \left[T(r,z) - \tilde{T}(r,z)\right]^2 dr dz = \min.\,, \tag{5.1}$$

where $n = n^1 + n^2$, S^i is the surface of one finite element, $T(r,z)$ is the temperature in the new mesh, $\tilde{T}(r,z)$ is the temperature given in the old mesh and n is the number of elements in the new mesh. Let us substitute (4.1) into equation (5.1), and transform it into the $[-1,1]$ interval, so that the numerical integration can be computed via Gauss quadrature.

$$\sum_{i=1}^{n} \int_{-1}^{1}\int_{-1}^{1} \left[N_j(\xi,\eta)T_{ij} - N_j(\tilde{\xi},\tilde{\eta})\tilde{T}_{\tilde{i}j}\right]^2 \left| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{array} \right| d\xi d\eta = \min.\,, \tag{5.2}$$

where the subscripts i and $\tilde{i}$ denote the elements in the new mesh and those elements in the old mesh, which coincide with the point with coordinates (ξ,η) in the new mesh, respectively. The subscript index j refers to the shape functions. Let T_l be a vector. The size of this vector equals the number of degrees of freedom of the problem. After the summation in (5.2) the components of T_{ij} are added to the appropriate element of T_l. This element of T_l can be determined by the connection between the local and global numbering [6]. After deriving equation (5.2) with respect to T_{ij} and making the summation for every element, the following linear system of equations is obtained:

$$\sum_{l=1}^{m} A_{kl}T_l = b_k\,, \tag{5.3}$$

where m is the number of degrees of freedom of the temperature field,

$$A_{kl} = \sum_{i=1}^{n} \int_{-1}^{1}\int_{-1}^{1} N_k(\xi,\eta)N_l(\xi,\eta) \left| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{array} \right| d\xi d\eta\,,$$

and

$$b_k = \sum_{i=1}^{n} \int_{-1}^{1}\int_{-1}^{1} N_k(\xi,\eta)N_j(\tilde{\xi},\tilde{\eta})\tilde{T}_{\tilde{i}j} \left| \begin{array}{cc} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{array} \right| d\xi d\eta\,.$$

From this equation the vector of temperature parameters T_l can be computed. The values of temperature have to be known in Gauss or Lobatto points (ξ,η) because of the numerical integration. However, the Gauss or Lobatto integration points in the new mesh and in the old mesh do not coincide. So the problem is to find the point P in the appropriate element $(\tilde{i})$ with local coordinates $(\tilde{\xi},\tilde{\eta})$ in the other mesh with practically the same contour – see Figure 1.

Instead of checking each element on whether it contains the point P or not (as in [10]), the searching process is realized by a new recursive algorithm. It is practical to create a list of the surrounding elements of each element, and a list of already checked elements [15]. Within a step one element is examined. If the element has already been checked, the step is finished. The local coordinates $(\tilde{\xi},\,\tilde{\eta})$ of the point P have to

be computed by the Newton-Raphson method. If the local coordinates are between -1 and 1, the algorithm is finished. If $\tilde{\xi}$ and $\tilde{\eta}$ are outside the interval $[-1, 1]$, the element is marked as "checked". After that, the neighbors of the element have to be checked. The neighbors can be determined with the help of the list of the surrounding elements. An order have to be defined among the neighbors. On ground of this order the search goes on as a new step. The order can be arbitrary, but the process is faster if a suitable order is chosen. A possible way is the following. Let R be a ray that starts from the point Q (see Figure 2), and passes through the elements to the point P. The ray crosses one of the edges of the starting element. The element next to

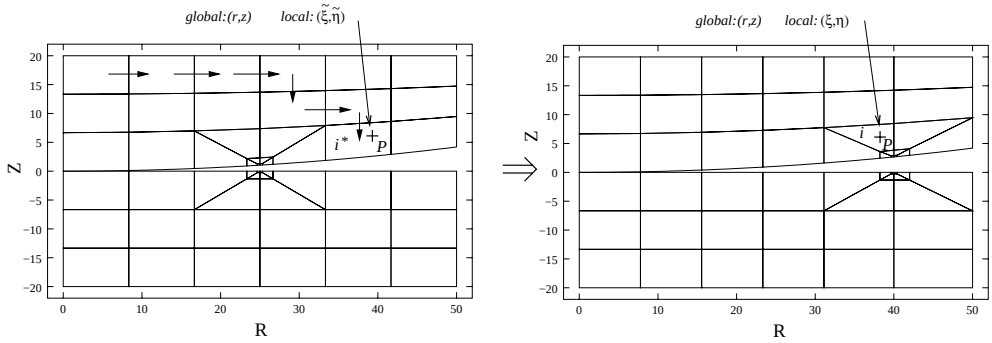

Figure 1. If element i and point (r, z) are known, the appropriate element i^* and local coordinate $(\tilde{\xi}, \tilde{\eta})$ are sought on the old mesh.

this edge will be the first in the order. If the ray passes through a node, an arbitrary neighboring edge of this node can be considered. With this algorithm in hand the element containing the point P can be rapidly found.

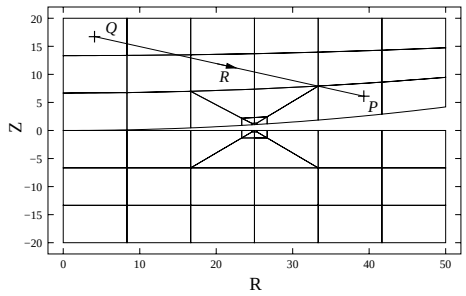

Figure 2. Let R be a ray that starts from the point Q, and passes through the elements to the point P. R crosses a side of the element containing Q, and enters the neighboring element. This neighboring element is the first in the further search process.

6. **Numerical examples**

Let us consider an axisymmetrical body with inner radius $R_i = 80$ mm and outer radius $R_o = 120$ mm. The height of the body can be considered infinite. In other words this is an infinitely long tube. The heat flux on the inner and outer surface equals zero, i.e., the body is thermally isolated. The initial temperature $f(r)$ is given. Because of its symmetry this problem can be treated with a one dimensional mathematical model. The following differential equation has to be solved:

$$c\rho\frac{\partial T(r,t)}{\partial t} = k\left(\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,t)}{\partial r}\right) \qquad R_i \le r \le R_o\,, \tag{6.1}$$

where the initial and boundary conditions are written as:

$$T(r,0) = f(r) \qquad R_i \le r \le R_o \tag{6.2}$$

$$-k\frac{\partial T(r,t)}{\partial r}\bigg|_{r=R_i} = 0 \qquad t \ge 0$$

$$-k\frac{\partial T(r,t)}{\partial r}\bigg|_{r=R_o} = 0 \qquad t \ge 0\,.$$

The specific heat is $c = 460$ J/kg℃, the material density is $\rho = 7850\text{kg/m}^3$, and the coefficient of heat conduction is $k = 55$ W/m℃. The exact solution of the problem is the following (see Appendix):

$$T_{ex}(r,t) = T_s + 2\sum_{n=1}^{\infty} C_n \frac{U_0(\omega_n r)}{R_o^2(U_0(\omega_n R_o))^2 - R_i^2(U_0(\omega_n R_i))^2} e^{-\kappa\omega_n t}\,, \tag{6.3}$$

where

$$U_0(\omega_n r) = J_0(\omega_n r)Y_1(\omega_n R_i) - Y_0(\omega_n r)J_1(\omega_n R_i),$$

$$C_n = \int_{R_i}^{R_o} (f(\xi) - T_s)\xi U_0(\omega_n \xi)d\xi,$$

$$T_s = \frac{2}{R_o^2 - R_i^2}\int_{R_i}^{R_o} \xi f(\xi)d\xi,$$

$\kappa = \dfrac{k}{c\rho}$, ω_n are the roots of the equation

$$U_1(\omega R_o) = J_1(\omega R_o)Y_1(\omega R_i) - Y_1(\omega R_o)N_1(\omega R_i) = 0$$

and $J_0(\xi)$, $J_1(\xi)$, $Y_0(\xi)$, $Y_1(\xi)$ are the first and second order Bessel functions of the first kind and the first and second order Bessel functions of the second kind, respectively [16]. One can see that the temperature field will be equalized if the bodies are thermally isolated. When the temperature gradient vanishes everywhere in the bodies, the temperature field will change no more. Accordingly, when the time becomes infinitely high in (6.3), the solution of the heat conduction problem will be the constant temperature.

Due to the symmetry of the problem, it is enough to solve the heat conduction problem between two parallel planes perpendicular to axis z. Let the two planes be

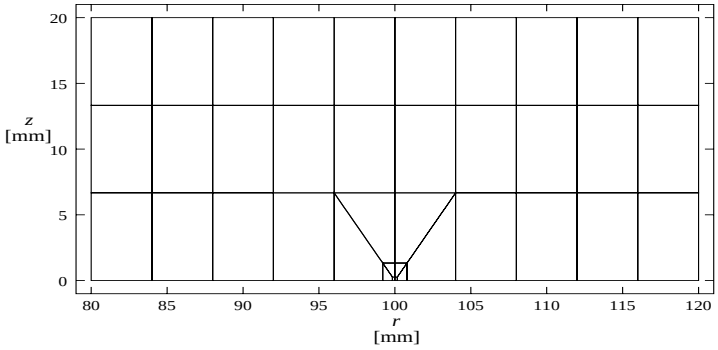

Figure 3. The cross-section of the body between $z_1 = 0$ mm and $z_2 = 20$ mm

situated at $z_1 = 0$ mm and at $z_2 = 20$ mm. The cross-section of the body bounded by the two planes is divided into finite elements – see Figure 3. The polynomial degree of the shape functions is 8, the number of degrees of freedom is 2522. The initial

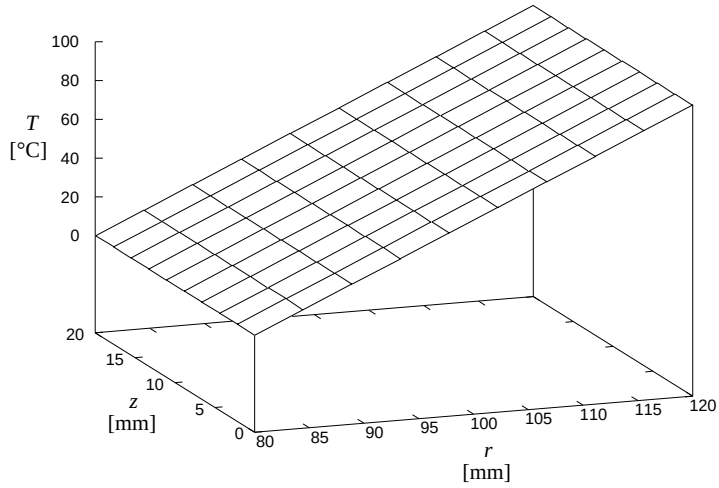

Figure 4. The initial temperature distribution for the first and second example

temperature is $T(r, 0) = f(r) = (2.5r - 200)°\mathrm{C}$ (see Figure 4) and the length of a time step is $\Delta t = 0.01$ s. To ensure the accuracy of six digits in the exact solution, the first 100 terms of the series are considered in (6.3).

The mesh is modified in every time step, so that the node, whereto the mesh is graded, moves from $r = 90$ mm to $r = 110$ mm. We will investigate the influence

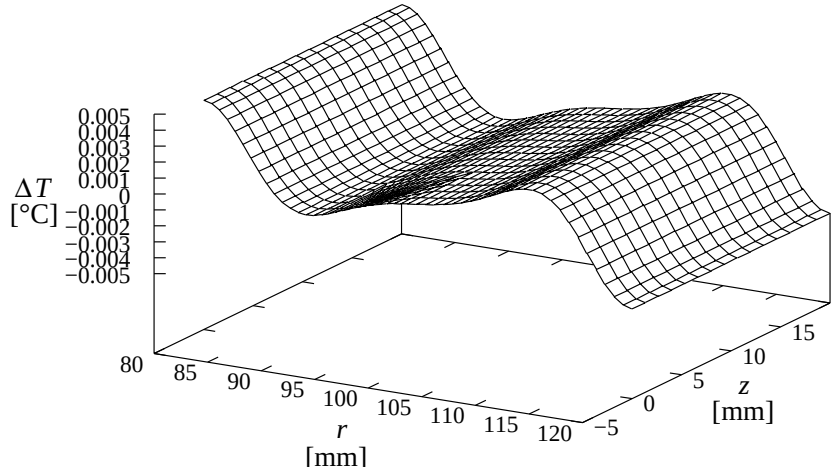

Figure 5. The difference between the exact solution and the finite element approximation

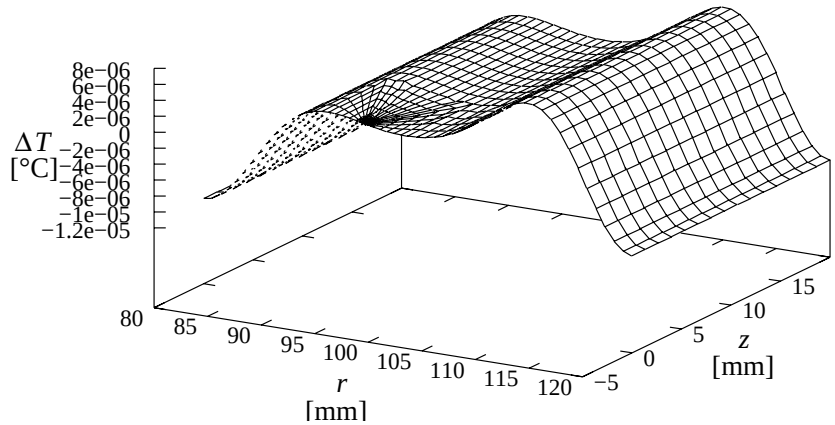

Figure 6. The difference between the finite element approximation on fixed mesh and finite element approximation on continuously changing mesh

of this modification on the finite element solution. The difference between the exact solution and the finite element approximation at $t = 1$ s can be seen in Figure 5. The relative error is solved by $e_1 = \Delta T_1/|T_{ex}|$, where $\Delta T_1 = |T_{ap} - T_{ex}|$, T_{ap} is the approximated temperature and T_{ex} is the exact temperature. The relative error is less than 0.01% everywhere in the bodies. The modification of the mesh has no significant influence on the accuracy of the finite element approximation. Now let us compare two finite element approximations. In the first case the mesh is fixed, and in the second case the mesh is modified in every time step, so that the node,

whereto the mesh is graded, moves from $r = 90$ mm to $r = 110$ mm. The boundary conditions are the same as in the first example. The time considered is $t = 1$ s, and the length of a time step is $\Delta t = 0.01$ s. One can see in Figure 6 that the difference between the two fields is small. The relative error is defined as $e_2 = \Delta T_2 / T_{ap}$, where $\Delta T_2 = |T_{am} - T_{ap}|$, T_{ap} is the approximated temperature in the fixed mesh and T_{am} is the approximated temperature in the modified mesh. The relative error is less than $1.2 \cdot 10^{-7}\%$, there is no significant difference between the two approximated temperature fields. After these two examples we can suppose that this technique

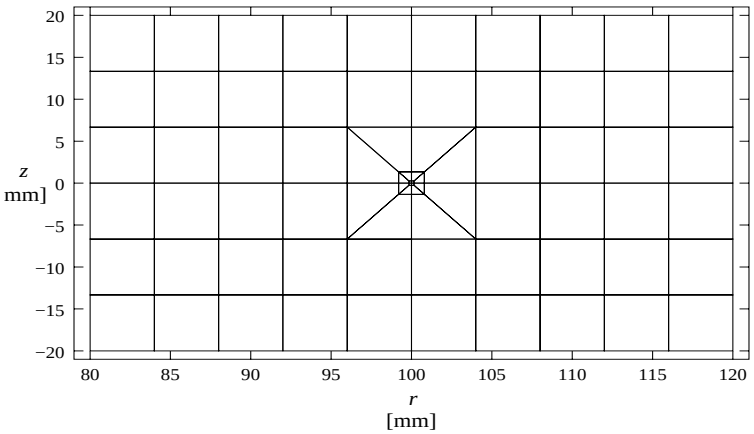

Figure 7. The finite element mesh of the cross-section of the bodies. The contact surface is the $z = 0$ plane. The boundaries of the contact region are the $r = 80$ mm edge and the point where the smallest element is situated, i.e., $r = 100$ mm

provides a reliable solution of the heat conduction problem in a continuously changing mesh. In the third example a possible application of this technique will be introduced. Let us consider a coupled thermo-mechanical contact problem. The system consists of two bodies, which slide on each other. The lower body is fixed, and the upper one is rotated with angular velocity $\omega = 1/\text{s}$. The inner and outer radii of the bodies are $R_i = 80$ mm and $R_o = 120$ mm, respectively, and their height is $h = 20$ mm. The contact surface is the $r\varphi$ plane – see Figure 7. The problem can be split into separately solved mechanical and thermo-dynamical problems. In this paper we disregard the computation of the contact problem. Instead a parabolic contact pressure distribution is assumed. The thermal boundary conditions are the same as equation (2.2). Let $\alpha^1 = \alpha^2 = 44$ $W/\text{m}^2\,^\circ\text{C}$, $c^1 = c^2 = 460$ J/kg°C, $\rho^1 = \rho^2 = 7850$ kg/m^3, $k^1 = k^2 = 55$ W/m°C, $\vartheta = 2/3$, $\Delta t = 0.01$ s, $c_D = 0.8$. The applied polynomial degree is 8. When the mesh is modified, the pressure distribution changes. The pressure can be written as: $p(r) = p_{max}\left[1 - (r - R_i)^2/(r_b - R_i)^2\right]$, where p_{max} is the maximum pressure, and r_b is the r coordinate of the node whereto the mesh is graded – see Figure 8. The initial temperature distribution is zero. It is supposed, that the boundary of the contact region, its radius is r_b, moves continuously from $r = 90$ mm

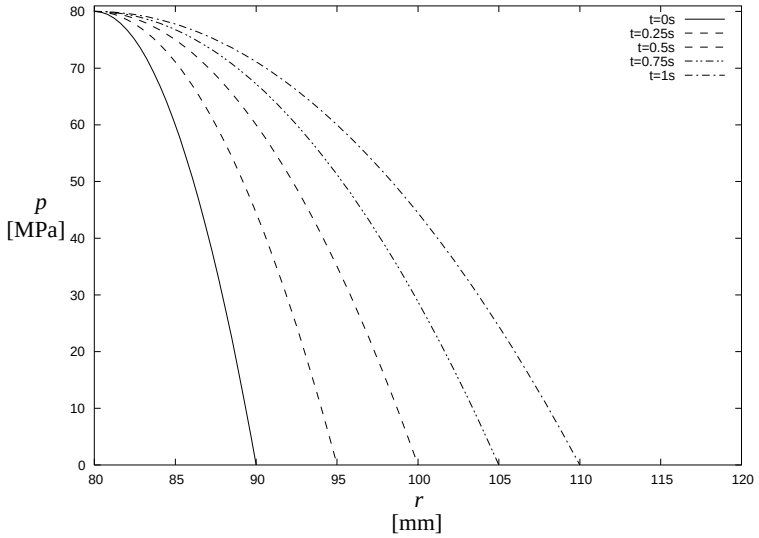

Figure 8. The contact pressure distribution at $t = 0$s, 0.25s, 0.5s, 0.75s, 1s

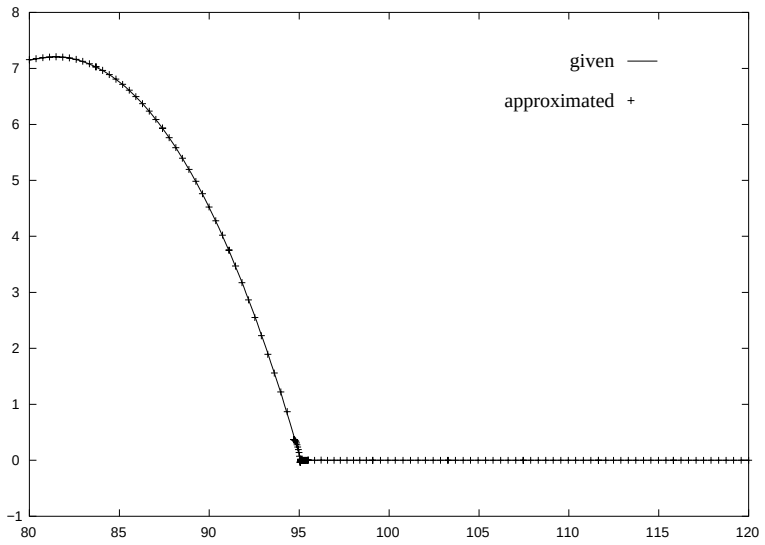

Figure 9. The given heat flux (see equation (2.2)) and the heat flux obtained from the approximated temperature field

to $r = 110$ mm during the process. One of the ways to verify the numerical solution is to check whether the boundary conditions meet the heat flux obtained from the

temperature field approximated: $q_{ap}^{e} = -k\dfrac{\partial T^{e}(r,t)}{\partial r}\Big|_{r\in\Gamma_c^e}$. Figure 9 shows that the derivative of the temperature field with respect to r gives accurately the boundary condition. The temperature distributions for the upper body at $t = 0$ s, $t = 0.25$ s, $t = 0.5$ s, $t = 0.75$ s and $t = 1$ s can be seen in Figures 10-14. Due to the fact, that

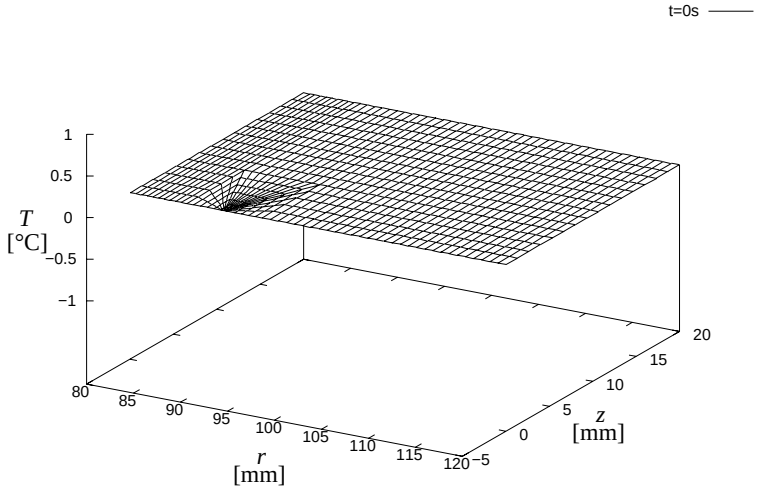

Figure 10. The approximated temperature distribution of the upper body at $t = 0$ s

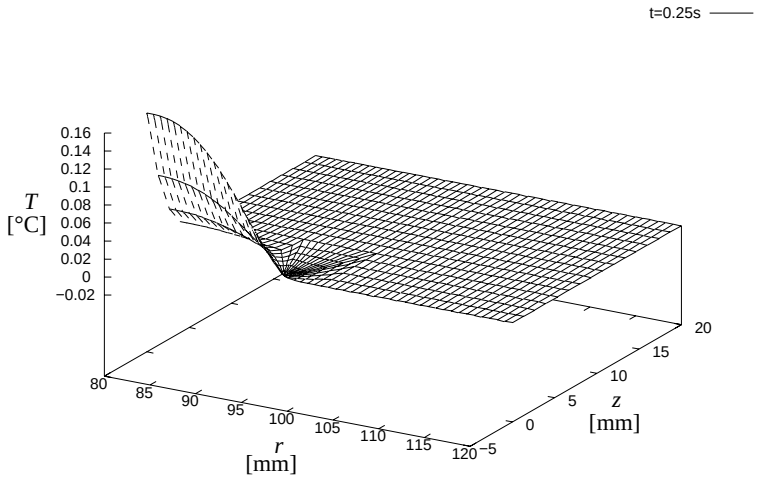

Figure 11. The approximated temperature distribution of the upper body at $t = 0.25$ s

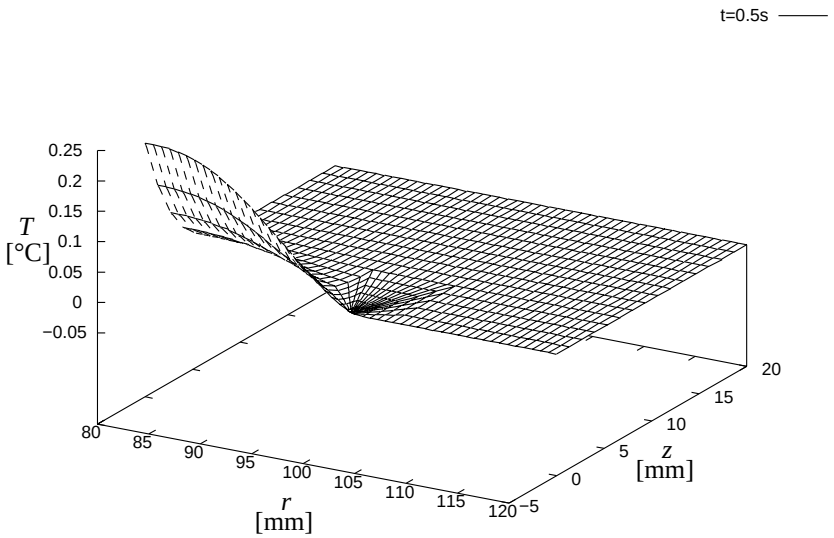

Figure 12. The approximated temperature distribution of the upper body at $t = 0.5$ s

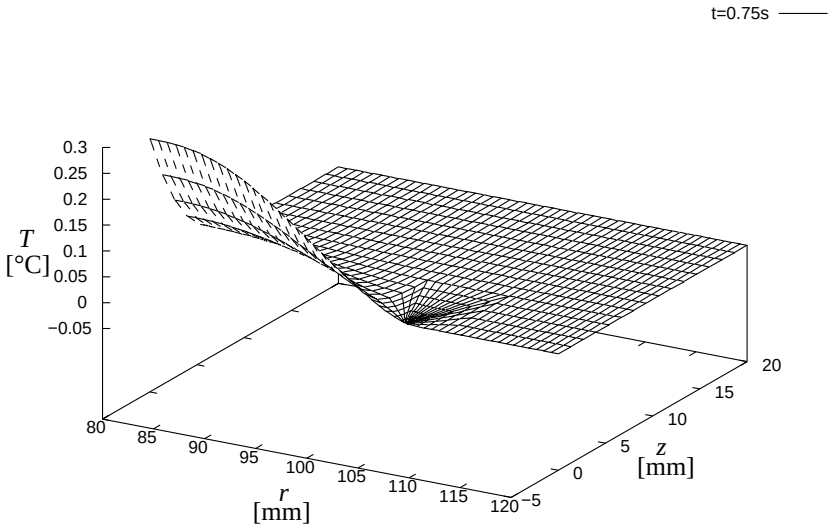

Figure 13. The approximated temperature distribution of the upper body at $t = 0.75$ s

the boundary conditions on the lower and upper bodies are the same, the temperature distributions of the lower and upper bodies are mirror images of each other.

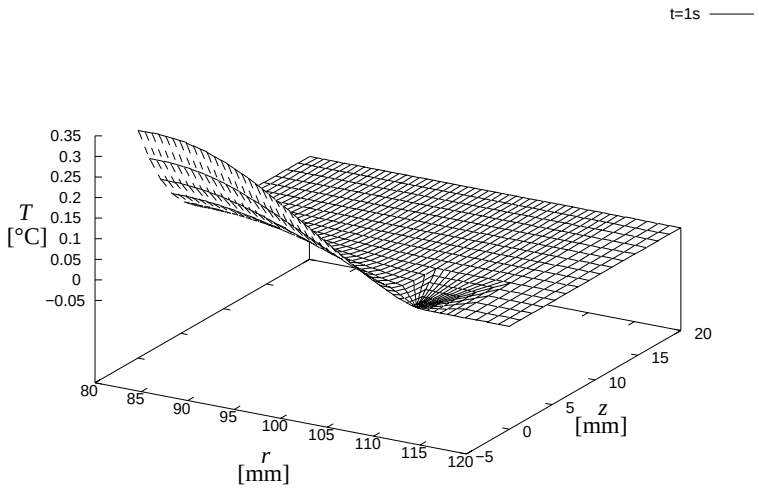

Figure 14. The approximated temperature distribution of the upper body at $t = 1$ s

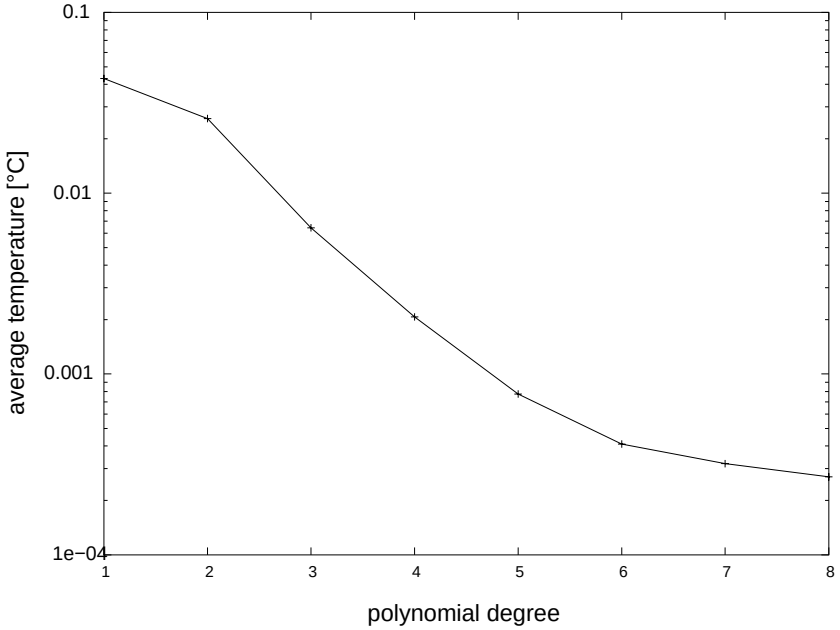

Figure 15. When the polynomial degree is increased, the error in average temperature is decreased

The convergence of the solution was also investigated with different polynomial degrees. At the end of the time interval the average temperature $\bar{T}$, which should be

proportional to the thermal energy, was computed

$$\bar{T} = \frac{1}{\Omega} \int_{\Omega} T(r,z) d\Omega \,. \tag{6.4}$$

This average temperature as a function of p is represented in Figure 15. When the polynomial degree is increased, the error in average temperature is decreased and the average temperature converges to a given value.

7. Conclusions

A special form of the *hp*-version of the finite element method has been presented for the solution of the heat conduction problem for axially symmetric bodies. The mesh was adjusted in every time step. The purpose was to simulate the thermal part of a thermo-mechanical contact problem. When the contact problem is solved, the mesh has to be adjusted so that the boundary of the contact zone is a nodal point. The high polynomial degree and the employment of small elements around the border of the contact region ensure high accuracy in the numerical solution. When the contact region changes the mesh has to be modified. Using the mapping technique, the new temperature field can be computed in the modified mesh with a great accuracy, and the heat conduction problem can be solved accurately. Exact solutions and finite element approximations were compared with each other to prove the reliability of the computation.

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APPENDIX

Consider the following partial differential equation:

$$\frac{\partial \vartheta(r,t)}{\partial t} = \kappa \left(\frac{\partial^2 \vartheta(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial \vartheta(r,t)}{\partial r} \right), \qquad \begin{matrix} r_1 \le r \le r_2 \\ 0 \le t \le t_{\max} \end{matrix} \tag{A.1}$$

$$\vartheta(r,0) = f(r) \tag{A.2}$$

$$k \frac{\partial \vartheta(r,t)}{\partial r}\bigg|_{r=r_1} = 0 \tag{A.3}$$

$$k \frac{\partial \vartheta(r,t)}{\partial r}\bigg|_{r=r_2} = 0 \tag{A.4}$$

where $\kappa = \dfrac{k}{\rho c}$, $f(r)$ is a given function, r_1 and r_2 are the inner and outer radii, respectively, and $t_{\max}$ is the length of the time interval considered. The solution is searched for as the sum of the solution of a steady-state problem and a solution of a transient problem: $\vartheta(r,t) = u(r) + v(r,t)$.

1. The steady-state problem:

$$\kappa \left(\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} \right) = 0, \quad r_1 \le r \le r_2 \tag{A.5}$$

$$k \frac{du(r)}{dr}\bigg|_{r=r_1} = 0 \tag{A.6}$$

$$k \frac{du(r)}{dr}\bigg|_{r=r_2} = 0 \tag{A.7}$$

As is well known, the general solution of equation (A.5) is given by the formula:

$$u(r) = c_1 \ln r + c_2 \,, \tag{A.8}$$

where c_1 can be determined from the boundary conditions:

$$\left.\begin{aligned}\left.\frac{du(r)}{dr}\right|_{r=r_1} &= \frac{c_1}{r_1} = 0\\ \left.\frac{du(r)}{dr}\right|_{r=r_2} &= \frac{c_1}{r_2} = 0\end{aligned}\right\} \quad \Rightarrow \quad c_1 = 0 \tag{A.9}$$

c_2 is determined later with the help of the solution of the transient problem. At this moment the solution of equation (A.5) is

$$u(r) = c_2 (= \text{const.}) \,. \tag{A.10}$$

2. The transient problem:

$$\frac{\partial v(r,t)}{\partial r} = \kappa \left(\frac{\partial^2 v(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,t)}{\partial r} \right), \qquad \begin{matrix} r_1 \le r \le r_2 \\ 0 \le t \le t_{\max} \end{matrix} \tag{A.11}$$

$$v(r,t)|_{t=0} = f(r) - u(r) \,, \tag{A.12}$$

$$k \left.\frac{\partial v(r,t)}{\partial r}\right|_{r=r_1} = 0 \,, \tag{A.13}$$

$$k \left.\frac{\partial v(r,t)}{\partial r}\right|_{r=r_2} = 0 \,. \tag{A.14}$$

Let us try to find the solution of equations (A.11-A.14) as the product of two functions:

$$v(r,t) = \varphi(r)\psi(t) \tag{A.15}$$

Substituting equation (A.15) into equation (A.11) and separating the variables r and t we obtain

$$\frac{1}{\kappa}\frac{1}{\psi(t)}\frac{d\psi(t)}{dt} = \frac{1}{\varphi(r)}\frac{d^2\varphi(r)}{dr^2} + \frac{1}{r\varphi(r)}\frac{d\varphi(r)}{dr} = \pm\omega^2. \tag{A.16}$$

Because the left and right hand sides are independent, they must equal to the same constant. It is worth giving this constant in the form: $\pm\omega^2$. With this in hand two ordinary differential equations have to be solved instead of a partially differential equation.

a.

$$\frac{d\psi(t)}{dt} = \pm\kappa\omega^2\psi(t) \,. \tag{A.17}$$

The solution of equation (A.17) has the form

$$\psi(t) = c_3 e^{\pm\kappa\omega^2 t} \,. \tag{A.18}$$

If the exponent is greater than zero for $t \to \infty$ then the function $\psi(t)$ is divergent. That is reason why only the negative sign is acceptable:

$$\psi(t) = c_3 e^{-\kappa\omega^2 t} \tag{A.19}$$

b.

$$r\frac{d^2\varphi(r)}{dr^2}+\frac{d\varphi(r)}{dr}+\omega^2 r\varphi(r)=0\,. \tag{A.20}$$

The solution of equation (A.20) assumes the form

$$\varphi(r)=c_4\mathrm{J}_0(\omega r)+c_5\mathrm{Y}_0(\omega r)\,, \tag{A.21}$$

where $\mathrm{J}_0(r)$ and $\mathrm{Y}_0(r)$ are the first order Bessel functions.

Let us substitute (A.19) and (A.21) into (A.15)

$$v(r,t)=e^{-\kappa\omega^2 t}(\underbrace{c_3c_4}_{c_6}\mathrm{J}_0(\omega r)+\underbrace{c_3c_5}_{c_7}\mathrm{Y}_0(\omega r)) \tag{A.22}$$

We can now substitute the solution (A.22) into the boundary condition (A.13)

$$-k\left.\frac{\partial v(r,t)}{\partial r}\right|_{r=r_1}=ke^{-\kappa\omega^2 t}(c_6\omega\mathrm{J}_1(\omega r_1)+c_7\omega\mathrm{Y}_1(\omega r))=0\,, \tag{A.23}$$

where $\mathrm{J}_1(r)$ and $\mathrm{Y}_1(r)$ are the second order Bessel functions. The boundary conditions can be satisfied only if $c_6=C\mathrm{Y}_1(\omega r)$ and $c_7=C\mathrm{J}_1(\omega r)$, where C is a constant. The following notations will be employed:

$$\mathrm{U}_0(\omega r):=\mathrm{Y}_1(\omega r_1)\mathrm{J}_0(\omega r)-\mathrm{J}_1(\omega r_1)\mathrm{Y}_0(\omega r) \tag{A.24}$$

and

$$\mathrm{U}_1(\omega r):=\mathrm{Y}_1(\omega r_1)\mathrm{J}_1(\omega r)-\mathrm{J}_1(\omega r_1)\mathrm{Y}_1(\omega r)\,. \tag{A.25}$$

It can be proved that

$$\frac{d\mathrm{U}_0(\omega r)}{dr}=-\omega\mathrm{U}_1(\omega r)\,. \tag{A.26}$$

With the help of equation (A.24) the solution of equation (A.11) can be written in the following form:

$$v(r,t)=Ce^{-\kappa\omega^2 t}\mathrm{U}_0(\omega r)\,. \tag{A.27}$$

Let us substitute equation (A.27) into the boundary condition (A.14)

$$-k\left.\frac{\partial v(r,t)}{\partial r}\right|_{r=r_2}=ke^{-\kappa\omega^2 t}\omega\mathrm{U}_1(\omega r_2)=0\,. \tag{A.28}$$

This means that the roots of $\mathrm{U}_1(\omega r)=0$ have to be determined to satisfy equation (A.14). Let us denote the roots by ω_n $(n=1,2,\dots)$. The functions $\mathrm{U}_0(\omega_n r)$ are independent, hence the solution of (A.11) can be written as a series

$$v(r,t)=\sum_{n=1}^{\infty}C_n e^{-\kappa\omega_n^2 t}\mathrm{U}_0(\omega_n r)\,. \tag{A.29}$$

Making use of the orthogonality of the Bessel functions one can prove the orthogonality condition

$$\int_{r_1}^{r_2} r\mathrm{U}_0(\omega_n r)\mathrm{U}_0(\omega_m r)dr=\begin{cases}\dfrac{1}{2}\left[r_1^2\mathrm{U}_0^2(\omega_n r_1)-r_2^2\mathrm{U}_0^2(\omega_n r_2)\right] & \text{if } n=m\\ 0 & \text{if } n\neq m\,.\end{cases} \tag{A.30}$$

Let us substitute series (A.29) into the boundary condition (A.12):

$$v(r,t)|_{t=0} = \sum_{n=1}^{\infty} C_n \mathrm{U}_0(\omega_n r) = f(r) - u(r)\,. \tag{A.31}$$

Let us multiply equation (A.31) by $r\mathrm{U}_0(\omega_m r)$ and integrate it on the $[r_1, r_2]$ interval

$$\int_{r_1}^{r_2} (f(r) - u(r))\mathrm{U}_0(\omega_m r) r dr = \sum_{n=1}^{\infty} C_n \int_{r_1}^{r_2} \mathrm{U}_0(\omega_n r)\mathrm{U}_0(\omega_m r) r\, dr\,. \tag{A.32}$$

The constant C_n can be determined with the aid of equation (A.30):

$$C_n = \frac{\displaystyle\int_{r_1}^{r_2} (f(r) - u(r))\mathrm{U}_0(\omega_n r) r dr}{\dfrac{1}{2}\left[r_1^2 \mathrm{U}_0^2(\omega_n r_1) - r_2^2 \mathrm{U}_0^2(\omega_n r_2)\right]}\,. \tag{A.33}$$

The solution of equation (A.11) is of the form

$$v(r,t) = \sum_{n=1}^{\infty} \frac{\displaystyle\int_{r_1}^{r_2} (f(r) - u(r))\mathrm{U}_0(\omega_n r) r dr}{\dfrac{1}{2}\left[r_1^2 \mathrm{U}_0^2(\omega_n r_1) - r_2^2 \mathrm{U}_0^2(\omega_n r_2)\right]} e^{-\kappa\omega_n^2 t}\mathrm{U}_0(\omega_n r) \tag{A.34}$$

Finally the constant c_2 in equation (A.10) has to be determined. Let us multiply (A.1) by r and integrate it over the interval $[r_1, r_2]$

$$\int_{r_1}^{r_2} \frac{\partial \vartheta(r,t)}{\partial t} r dr = \int_{r_1}^{r_2} \kappa \left[\frac{\partial^2 \vartheta(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial \vartheta(r,t)}{\partial r}\right] r dr\,. \tag{A.35}$$

It can be easily proved that

$$\left[\frac{\partial^2 \vartheta(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial \vartheta(r,t)}{\partial r}\right] r = \frac{\partial}{\partial r}\left(r\frac{\partial \vartheta(r,t)}{\partial r}\right)\,. \tag{A.36}$$

With the aid of (A.36), the boundary conditions (A.3) and (A.4) and the equation $\kappa = k/\rho c$ we can calculate the right side of equation (A.35)

$$\int_{r_1}^{r_2} \frac{\partial \vartheta(r,t)}{\partial t} r dr = r \underbrace{\kappa \frac{\partial \vartheta(r,t)}{\partial r}\bigg|_{r=r_2}}_{=0} - r \underbrace{\kappa \frac{\partial \vartheta(r,t)}{\partial r}\bigg|_{r=r_1}}_{=0} = 0 \tag{A.37}$$

or

$$\frac{\partial}{\partial t}\int_{r_1}^{r_2} \vartheta(r,t) r dr = 0 \quad \Rightarrow \quad \int_{r_1}^{r_2} \vartheta(r,t) r dr = \text{constant}\,. \tag{A.38}$$

The temperature distribution $\vartheta(r,t)$ is decomposed into a steady-state and a transient temperature distributions

$$\int_{r_1}^{r_2} \underbrace{\vartheta(r,0)}_{f(r)} r dr = \int_{r_1}^{r_2} \vartheta(r,\infty) r dr = \int_{r_1}^{r_2} \underbrace{u(r)}_{c_2} r dr + \underbrace{\int_{r_1}^{r_2} v(r,\infty) r dr}_{=0} = c_2(r_2^2 - r_1^2)\,. \quad \text{(A.39)}$$

The constant c_2 follows from equation (A.39)

$$c_2 = \frac{1}{r_2^2 - r_1^2} \int_{r_1}^{r_2} f(r) r dr \quad \text{(A.40)}$$

Making use of the foregoing we can write

$$\vartheta(r,t) = \frac{1}{r_2^2 - r_1^2} \int_{r_1}^{r_2} f(\zeta)\zeta d\zeta + \\ + \sum_{n=1}^{\infty} \frac{\displaystyle\int_{r_1}^{r_2} \left(f(\eta) - \frac{1}{r_2^2 - r_1^2} \int_{r_1}^{r_2} f(\xi)\xi d\xi \right) \mathrm{U}_0(\omega_n \eta)\eta d\eta}{\dfrac{1}{2}\left[r_1^2 \mathrm{U}_0^2(\omega_n r_1) - r_2^2 \mathrm{U}_0^2(\omega_n r_2) \right]} e^{-\kappa\omega_n^2 t} \mathrm{U}_0(\omega_n r)$$

which gives the temperature distribution we wanted to determine.