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CALCULATING TURBULENT FLOWS BASED ON A STOCHASTIC MODEL

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Abstract. Relying on his theoretical and experimental examinations, O. Reynolds arrived at the conclusion that the Navier-Stokes equation of motion describing laminar flow continues to remain valid in terms of velocity fields interpreted by instantaneous values when the fluid is in turbulent motion. Th. von Kármán also used experimental experience to construct his similarity hypothesis [1], according to which, on the one hand, outside the viscous layer close to the wall, the turbulent velocity distribution does not depend on the viscosity of the medium, and on the other, the local (turbulent) flow patterns show mechanical similarities in points of the fully developed turbulent flow field; in other words each of them can be transferred into a common (turbulent) flow pattern by means of a suitably chosen transformation.

The turbulence model used in this paper also relies on the same hypothesis. The model's fundamental principle can be summed up in the following words: in any point of the flow field the Helmholz-Thomson vortex theorem valid in the relative coordinate system - that is a coordinate system moving steadily at a velocity equal to the average Reynolds velocity in the given point - is suitable for describing the turbulent velocity fluctuation, thus it can be considered to be the equation of motion of turbulence, which then can be transformed, on the basis of von Kármán's similarity hypothesis, into the coordinate system of the common flow pattern mentioned. And a particular solution to the partial differential equation obtained can be used to represent the stochastic flow of the turbulence while the optional coefficients and phase constants appearing in it as integration constants are considered to be probability variables. By using the scalar components of the turbulent velocity fluctuation obtained in this way – in this special relative coordinate system – it is possible to produce the scalar elements of Reynolds' turbulent stress tensor, which can be re-transformed into the physical space on the basis of mechanical similarity. Thus, by using the stochastic turbulence model it becomes possible to produce Reynolds' turbulent stress tensor in a specific way, which in the transport equations of turbulent motion leads to formal changes that can be used in the numerical solutions with advantage.

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1. Governing equations of turbulent motion

The turbulent motion of a fluid continuum in the Eulerian approach can be described by the superposition of two velocity fields depending on place and time: one is velocity fluctuation changing rapidly and in a stochastic way in time, and the other velocity field determines the motion of the fluid continuum without fluctuations but showing changes in time. The latter can naturally be a motion constant with time, when we have to do with the steady flow of the continuum; by contrast the velocity field describing turbulent fluctuation is invariably a function of time. Each characteristic of the instantaneous state of motion (scalar, vector or tensor) can be considered to be composed of two components: instantaneous value = mean value + fluctuation. The mean value – i.e. the Reynolds time average – is an integral mean value of a given motion characteristic referring to a time interval t_0 that is relatively big compared to the period of turbulent fluctuation:

$$f(\mathbf{x},t)=\frac{1}{t_0}\int\limits_{t}^{t+t_0} f_T(\mathbf{x},\tau)d\tau \ , \tag{1.1}$$

where $\mathbf{x}$ is the vector of place, t is time; the function $f(x,t)$ may interpret a scalar, vector or tensor field; and the subscript T here refers to 'instantaneous value'. The Reynolds time-mean of the fluctuation is naturally zero. Accordingly, the instantaneous velocity field of the fluid continuum performing the turbulent motion can be written in the form

$$\mathbf{v}_T(\mathbf{x},t)=\mathbf{v}(\mathbf{x},t)+\mathbf{v}'(\mathbf{x},t) \ , \tag{1.2}$$

where $\mathbf{v}(\mathbf{x},t)$ is the Reynolds mean velocity field and $\mathbf{v}'(\mathbf{x},t)$ is the velocity field of turbulent fluctuation. The curl of the velocity leads to the instantaneous vortex field, which is also the sum of two components, the Reynolds mean vortex field and the turbulent fluctuation field:

$$\boldsymbol{\Omega}_T(\mathbf{x},t)=\nabla\times\mathbf{v}_T=\boldsymbol{\Omega}(\mathbf{x},t)+\boldsymbol{\Omega'}(\mathbf{x},t) \ . \tag{1.3}$$

The surface forces (caused by molecular viscosity) arising between the fluid particles are also subject to stochastic fluctuation in the instantaneous velocity field of the turbulent flow; i.e. the instantaneous value of the tensor field expressing the stresses arising on the surface of the fluid components can be given similarly in the form

$$\boldsymbol{F}_T(\mathbf{x},t)=\boldsymbol{F}(\mathbf{x},t)+\boldsymbol{F}'(\mathbf{x},t) \ , \tag{1.4}$$

where tensor $\boldsymbol{F}(\mathbf{x},t)$ expresses the Reynolds' time-mean value, and tensor $\boldsymbol{F}'(\mathbf{x},t)$ expresses the fluctuation caused by the turbulence; the time-mean value of the latter being also zero. With respect to the fact that pressure in a viscous fluid in motion is equal to the negative of the first scalar invariant of the stress tensor, the following relationship:

$$p_T=-\frac{1}{3}(F_{11}+F_{22}+F_{33})-\frac{1}{3}(F'_{11}+F'_{22}+F'_{33})=p+p' \ ,$$

where p' is turbulent pressure fluctuation, and p is thermodynamic pressure (the pressure in the thermodynamic equation of state of the medium in flow), holds for the instantaneous value of pressure in a turbulent flow as follows from equation (1.4). Accordingly, the Reynolds time-mean of instantaneous turbulent pressure is equal to thermodynamic pressure. In our investigation we tend to suppose that the Stokes molecular viscosity law also holds for the instantaneous turbulent motion of a viscous fluid:

$$\boldsymbol{F}_T = -p_T \boldsymbol{I} + \eta \left[(\mathbf{v}_T \circ \nabla + \nabla \circ \mathbf{v}_T) - \frac{2}{3} (\nabla \cdot \mathbf{v}_T) \boldsymbol{I} \right] , \tag{1.5}$$

where η is the dynamic viscosity factor (considered constant), and $\boldsymbol{I}$ is unit tensor. Assuming that, on the one hand, the law of the conservation of mass continues to hold for the turbulent motion of the fluid continua, and that, on the other, the Navier-Stokes equation of motion gives an appropriate description of the instantaneous motion as well, in the velocity field of turbulent flow interpreted by means of instantaneous values, the equation of continuity expressing the conservation of mass can be written in the form

$$\frac{\partial \rho_T}{\partial t} + \nabla \cdot \rho_T \mathbf{v}_T = 0 \tag{1.6}$$

and the Navier-Stokes equation of motion can be given in the form

$$\rho_T \frac{\partial \mathbf{v}_T}{\partial t} + \rho_T (\mathbf{v}_T \cdot \nabla) \mathbf{v}_T = \rho_T \mathbf{g} + \mathrm{Div}\ \boldsymbol{F}_T , \tag{1.7}$$

where $\mathbf{g}$ is the specific value of the field force referring to a mass unit. If the medium in flow is incompressible ($\rho_T = \rho = \mathrm{const}$), the equation of continuity takes the form $\nabla \cdot \mathbf{v}_T = 0$, from which it follows that for an incompressible medium the turbulent fluctuation is $\nabla \cdot \mathbf{v}' = 0$ also for the velocity field of $\mathbf{v}'(\mathbf{r}, t)$. Now, substituting the expression $\mathbf{v}_T = \mathbf{v} + \mathbf{v}'$ into the above equations and taking the time-mean value of each member, in the $\mathbf{v}(\mathbf{r}, t)$ Reynolds mean velocity field the equation of continuity and the equation of motion take the following forms:

$$\nabla \cdot \mathbf{v} = 0 \tag{1.8}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \rho \mathbf{g} - \nabla p + \eta \Delta \mathbf{v} + \mathrm{Div}\ \boldsymbol{F}_R , \tag{1.9}$$

respectively, where $\boldsymbol{F}_R$ is the Reynolds turbulent stress tensor:

$$\boldsymbol{F}_R = -\rho \overline{(\mathbf{v}' \circ \mathbf{v}')} . \tag{1.10}$$

This tensor expresses a surface force in the Reynolds equation of motion (1.9) in the same way as the viscous stress tensor does, which, however, originates not from the molecular viscosity of the fluid, but from the change of momentum due to turbulent velocity fluctuation.

It should be noted that for the laminar flow of an incompressible medium developing in a potential field of force ($\mathbf{g} = -\nabla U$; U being the potential of the field of force) the equation of motion (1.9) will take the form

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\rho \nabla U - \nabla p + \eta \Delta \mathbf{v} \tag{1.11}$$

of the Navier-Stokes equation of motion, which, when supplemented with the equation of continuity $\nabla \cdot \mathbf{v} = 0$, gives a system of equations (consisting of a total of four scalar differential equations) for the purpose of determining the four unknown functions (the three components of velocity and the pressure). In the laminar case the three scalar equations of motion and the equation of continuity thus form a closed system of equations for determining the four unknown functions. In the case of turbulent flow, however, the equation of motion (1.9) – since the turbulent stress tensor $\boldsymbol{F}_R$ introduces new unknowns into it – does not form a closed system with the equation of continuity (1.8) any more.

In order to develop the equation of equilibrium of internal energy, our starting point is the theorem of the conservation of energy referring to transport processes, which is written for an incompressible medium in the velocity field of turbulent flow interpreted by instantaneous values for the case without dissipation heat transfer:

$$\rho c_P \left(\frac{\partial T_T}{\partial t} + (\mathbf{v}_T \cdot \nabla) \, T_T \right) = \nabla \cdot (\lambda \nabla T_T) + (\boldsymbol{\sigma}_T \cdot \nabla) \cdot \mathbf{v}_T \ , \tag{1.12}$$

where c_P is the specific heat of the medium in flow under constant pressure, λ is its thermal conduction coefficient, and $\boldsymbol{\sigma}_T$ is the deviator of the turbulent stress tensor $\boldsymbol{F}_T$ interpreted by instantaneous values, for which the Stokes formula of the form

$$\boldsymbol{\sigma}_T = \eta \ (\mathbf{v}_T \circ \nabla + \nabla \circ \mathbf{v}_T)$$

holds. Substituting relationship (1.2) into the above equation of the conservation of energy, and then taking the time-mean value of each member, a short calculation gives the scalar equation of the form

$$\rho c_P \left(\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) \, T \right) = \nabla \cdot (\lambda \nabla T) - \rho c_P \nabla \cdot (\overline{\mathbf{v}' T'}) + \rho(\varphi + \epsilon) \ , \tag{1.13}$$

where the first member on the right-hand side is an expression of molecular heat transport and the second one is that of turbulent heat transport; the third and the fourth member gives direct (viscous) dissipation and turbulent dissipation, resp.:

$$\varphi = \nu \ (\mathbf{v} \circ \nabla) \colon (\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v}) \quad ; \quad \epsilon = \nu \overline{(\mathbf{v}' \circ \nabla) : (\mathbf{v}' \circ \nabla + \nabla \circ \mathbf{v}')} \ , \tag{1.14}$$

where ν is the kinematic viscosity factor ($\nu = \eta / \rho$).

By way of summary of the above it can be stated that from the point of the solubility of a problem of non-isothermal turbulent flow three equations – the equation of continuity (1.8), the Reynolds equation of motion (1.9) and the energy equation (1.13) – are already available to determine the unknown velocity distribution $\mathbf{v}(\mathbf{x}, t)$, the pressure distribution $p(\mathbf{x}, t)$ and the temperature distribution $T(\mathbf{x}, t)$, but further unknown functions also appear in the equations: a) in equation (1.9) the turbulent stress tensor $\boldsymbol{F}_R$, b) in equation (1.13) the turbulent dissipation ϵ as well as the velocity-temperature correlation $\overline{\mathbf{v}' T'}$. Thus the number of unknowns is higher than the number of scalar differential equations at our disposal for solving them; i.e. the system of differential equations to be solved is underdetermined. It follows that in order to solve a given problem of turbulent flow, it is necessary to supplement the system of equations constituted by the equation of continuity, the Reynolds equation of motion and the equation of equilibrium of internal energy with further equations.

In order to develop the equation of equilibrium of the specific turbulent kinetic energy defined by equation $k = \overline{\mathbf{v}' \cdot \mathbf{v}'}/2$, the Navier-Stokes equation (1.7) of the fluid continuum moving in the velocity field $\mathbf{v}_T(\mathbf{x}, t)$ interpreted by instantaneous values is first multiplied scalarly by the velocity vector $\mathbf{v}'$ of turbulent fluctuation, then its time-mean value is taken, and then after ordering the following expression

$$\frac{dk}{dt}+\left(\overline{\mathbf{v}'\circ\mathbf{v}'}\right):(\nabla\circ\mathbf{v})+\overline{\mathbf{v}'\cdot\nabla\frac{\mathbf{v}'\cdot\mathbf{v}'}{2}}+\frac{1}{\rho}\overline{\mathbf{v}'\cdot\nabla p'}-\nu\overline{\mathbf{v}'\cdot\Delta\mathbf{v}'}=0 \qquad (1.15)$$

is obtained. Let us now take into account that for an incompressible fluid the following two equalities

$$\nabla\left[\overline{\mathbf{v}'\left(\frac{\mathbf{v}'\cdot\mathbf{v}'}{2}+\frac{p'}{\rho}\right)}\right]=\overline{\mathbf{v}'\cdot\nabla\frac{\mathbf{v}'\cdot\mathbf{v}'}{2}}+\frac{1}{\rho}\overline{\mathbf{v}'\cdot\nabla p'} \qquad (1.16)$$

$$\Delta\frac{\mathbf{v}'\cdot\mathbf{v}'}{2}=\mathbf{v}'\cdot\Delta\mathbf{v}'+(\mathbf{v}'\circ\nabla):(\mathbf{v}'\circ\nabla+\nabla\circ\mathbf{v}')-(\mathbf{v}'\circ\nabla):(\nabla\circ\mathbf{v}')$$

hold. After ordering and taking the time-mean value, the latter gives the following equation

$$\overline{\mathbf{v}'\cdot\Delta\ \mathbf{v}'}=\Delta\frac{\overline{\mathbf{v}'\cdot\mathbf{v}'}}{2}-\overline{(\mathbf{v}'\circ\nabla):(\mathbf{v}'\circ\nabla+\nabla\circ\mathbf{v}')}+\overline{(\mathbf{v}'\circ\nabla):(\nabla\circ\mathbf{v}')}\,,$$

which, multiplied by the kinematic viscosity factor ν and taking into account the definition equation (1.14) of the turbulent dissipation ϵ, results in the following:

$$\nu\overline{\mathbf{v}'\cdot\Delta\ \mathbf{v}'}=\nu\Delta k-\epsilon+\nu\overline{(\mathbf{v}'\circ\nabla):(\nabla\circ\mathbf{v}')}\,.$$

Let us now determine first the divergence of tensor $\mathbf{v}'\circ\mathbf{v}'$:

$$\mathrm{Div}\ (\mathbf{v}'\circ\mathbf{v}')=(\mathbf{v}'\circ\mathbf{v}')\cdot\nabla=(\mathbf{v}'\cdot\nabla)\,\mathbf{v}'$$

and then also the divergence of the vector obtained, and take the time-mean value of the latter:

$$\nabla\cdot\left[\mathrm{Div}\ \left(\overline{\mathbf{v}'\circ\mathbf{v}'}\right)\right]=\overline{(\mathbf{v}'\circ\nabla):(\nabla\circ\mathbf{v}')}\,.$$

Using the former relations gives the following:

$$\nu\overline{\mathbf{v}'\cdot\Delta\ \mathbf{v}'}=\nu\nabla\cdot\left[\nabla k+\mathrm{Div}\ \left(\overline{\mathbf{v}'\circ\mathbf{v}'}\right)\right]-\epsilon \qquad (1.17)$$

for the fifth term of equation (1.15). Substituting the relations (1.16) and (1.17) into equation (1.15) gives the equation of equilibrium of turbulent kinetic energy in the form

$$\frac{dk}{dt}+\left(\overline{\mathbf{v}'\circ\mathbf{v}'}\right):(\nabla\circ\mathbf{v})+\epsilon+\nabla\cdot\left[\overline{\mathbf{v}'\left(\frac{\mathbf{v}'\cdot\mathbf{v}'}{2}+\frac{p'}{\rho}\right)}-\nu\nabla k-\nu\ \mathrm{Div}\ \left(\overline{\mathbf{v}'\circ\mathbf{v}'}\right)\right]=0\,, \qquad (1.18)$$

which can be used as the supplementary equation of the system of equations constituted by the equation of continuity (1.8), the equation of motion (1.9) and the equation of equilibrium of internal energy (1.13).

2. Vortex theorems in turbulent flow

The German H. Ertel and the Russian A. A. Friedmann [2] showed independently of each other that if in a fluid continuum moving in a given velocity field $\mathbf{v}(\mathbf{x},t)$ there is also a second arbitrary vector field $\mathbf{a}(\mathbf{x},t)$ given, then the necessary and sufficient condition that the vector lines determined by the equation $\mathbf{a} \times d\mathbf{x} = \mathbf{0}$ should be constituted during the complete time of the motion by the same fluid particles and that the intensity $\mathbf{a} \cdot d\mathbf{A} = a\, dA_n$ of the elementary vector tubes constituted by the vector lines should remain unchanged is that the vector function $\mathbf{a}(\mathbf{x},t)$ should satisfy the requirement of the following equation:

$$\frac{\partial \mathbf{a}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{a} - (\mathbf{a} \cdot \nabla)\,\mathbf{v} + \mathbf{a}\,(\nabla \cdot \mathbf{v}) = \mathbf{0}\,. \tag{2.1}$$

After ordering and taking the rotations of all the members of the equation of motion (1.7) of the motion of the fluid developing in the turbulent velocity field $\mathbf{v}_T(\mathbf{x},t)$ interpreted by instantaneous values, in case of a potential force field gives the equation of the form

$$\frac{\partial \mathbf{\Omega}_T}{\partial t} + (\mathbf{v}_T \cdot \nabla)\,\mathbf{\Omega}_T - (\mathbf{\Omega}_T \cdot \nabla)\,\mathbf{v}_T + \mathbf{\Omega}_T\,(\nabla \cdot \mathbf{v}_T) = \nu \Delta \mathbf{\Omega}_T\,, \tag{2.2}$$

which is nothing else but the Helmholz-Thomson vortex theorem referring to the turbulent vortex field $\mathbf{\Omega}_T(\mathbf{x},t)$ interpreted by instantaneous values. In line with the Ertel-Friedmann theorem of vector-line conservation (2.1), thus in case of the turbulent flow of a compressible barotrope medium in a potential force field the instantaneous vortex lines determined by the equation $\mathbf{\Omega}_T \times d\mathbf{x} = \mathbf{0}$ do not remain unchanged, but are diffused into the surroundings, however, they remain unchanged, while ν tends to 0.

The vortex theorem concerning the velocity field $\mathbf{v}(\mathbf{x},t)$ interpreted by turbulent mean values is obtained from the vortex theorem (2.2) concerning the velocity field $\mathbf{v}_T(\mathbf{x},t)$ interpreted by instantaneous values by substituting $\mathbf{\Omega}_T = \mathbf{\Omega} + \mathbf{\Omega'}$ and taking the time-mean values:

$$\frac{\partial \mathbf{\Omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{\Omega} - (\mathbf{\Omega} \cdot \nabla)\,\mathbf{v} + \mathbf{\Omega}\,(\nabla \cdot \mathbf{v}) = \nu \Delta \mathbf{\Omega} + \nabla \times \overline{(\mathbf{v}' \times \mathbf{\Omega'})}\,. \tag{2.3}$$

Comparing this now with Ertel-Friedmann's theorem of conservation (2.1), it can be stated that in the velocity field interpreted by the turbulent mean velocity the vortex lines determined by the differential equation $\mathbf{\Omega} \times d\mathbf{x} = \mathbf{0}$ do not remain unchanged with the limit of ν tending to 0, but become diffused in the surroundings during the motion. The extent of vortex diffusion is determined by the expression on the right-hand side of equation (2.3): the first member is the viscosity of the medium in flow and the second one expresses the extent of vortex diffusion caused by turbulent exchange of momentum.

The vortex theorem regarding the velocity field of turbulent fluctuation $\mathbf{v}'(\mathbf{x},t)$ is obtained by the following train of thought. Let us consider the turbulent motion of a compressible barotrope fluid continuum in the surroundings of an arbitrarily fixed point P in the field. Let the turbulent mean velocity in the point be $\mathbf{v}_P$ and the vortex vector $\mathbf{\Omega}_P = \nabla \times \mathbf{v}_P$. In the surroundings of the fixed point P in an arbitrary running point Q

$$\mathbf{v}_T = \mathbf{v}_Q + \mathbf{v}' \quad ; \quad \boldsymbol{\Omega}_T = \nabla \times \mathbf{v}_T = \boldsymbol{\Omega}_Q + \boldsymbol{\Omega}' \tag{2.4}$$

are the turbulent instantaneous values. Since the extent of change in the time-mean velocity is by several orders smaller than that of turbulent fluctuation, the general validity is not restricted if – in the examination of the velocity field of turbulent fluctuation – the turbulent mean velocity field is considered to be steady (stationary) and thus velocity $\mathbf{v}_P$ and the vortex vector $\boldsymbol{\Omega}_P$ are regarded as constant. The vortex theorem (2.2) concerning the instantaneous turbulent velocity field is evidently valid also when seen from the relative coordinate system, whose origin performs a steady motion at the same speed as velocity $\mathbf{v}_P$ dominant in point P. In other words, it is possible to write the vortex theorem of the form

$$\frac{\partial \boldsymbol{\Omega}_T}{\partial t} + \left[(\mathbf{v}_T - \mathbf{v}_P)\cdot\nabla\right]\boldsymbol{\Omega}_T - (\boldsymbol{\Omega}_T\cdot\nabla)(\mathbf{v}_T - \mathbf{v}_P) + \boldsymbol{\Omega}_T\left[\nabla\cdot(\mathbf{v}_T - \mathbf{v}_P)\right] = \nu\Delta\boldsymbol{\Omega}_T$$

for the vortex field $\boldsymbol{\Omega}_T$ in the relative velocity field $\mathbf{v}_T - \mathbf{v}_P$. Let us now substitute the relationships (2.4) into this equation and then take boundary transition Q $\to$ P together with $\mathbf{v}_Q \to \mathbf{v}_P$ and $\boldsymbol{\Omega}_Q \to \boldsymbol{\Omega}_P$:

$$\begin{aligned}\frac{\partial}{\partial t}(\boldsymbol{\Omega}_P + \boldsymbol{\Omega}') + (\mathbf{v}'\cdot\nabla)(\boldsymbol{\Omega}_P + \boldsymbol{\Omega}') - \left[(\boldsymbol{\Omega}_P + \boldsymbol{\Omega}')\cdot\nabla\right]\mathbf{v}' + \\ + (\boldsymbol{\Omega}_P + \boldsymbol{\Omega}')\,\nabla\cdot\mathbf{v}' = \nu\Delta(\boldsymbol{\Omega}_P + \boldsymbol{\Omega}')\ .\end{aligned}$$

However, the vortex vector $\boldsymbol{\Omega}_P$ is constant in this equation, and therefore its derivatives (both according to time and to place) disappear. And since it can no longer lead to misunderstandings, subscript P can also be omitted from the vortex vector, and thus the former equation gives the vortex theorem of the form

$$\frac{\partial \boldsymbol{\Omega}'}{\partial t} + (\mathbf{v}'\cdot\nabla)\boldsymbol{\Omega}' - (\boldsymbol{\Omega}'\cdot\nabla)\mathbf{v}' + \boldsymbol{\Omega}'(\nabla\cdot\mathbf{v}') = (\boldsymbol{\Omega}\cdot\nabla)\mathbf{v}' - \boldsymbol{\Omega}(\nabla\cdot\mathbf{v}') + \nu\Delta\boldsymbol{\Omega}' \tag{2.5}$$

in the velocity field $\mathbf{v}'(\mathbf{x}, t)$ of turbulent fluctuation. Accordingly, the vortex lines determined by the equation $\boldsymbol{\Omega}' \times d\mathbf{x} = \mathbf{0}$ in the velocity field of turbulent fluctuation do not remain unchanged in the case $\nu \to 0$ either, but get diffused during the motion into the surroundings. With respect to the fact that, the higher Reynolds number the flow has (the better developed the turbulence is), the easier it is to neglect the intensity of viscous vortex diffusion as compared to that of turbulent vortex diffusion, the third term on the right-hand side of the above vortex theorem can be neglected, and the equation

$$\frac{\partial \boldsymbol{\Omega}'}{\partial t} + (\mathbf{v}'\cdot\nabla)\boldsymbol{\Omega}' - (\boldsymbol{\Omega}'\cdot\nabla)\mathbf{v}' + \boldsymbol{\Omega}'(\nabla\cdot\mathbf{v}') = (\boldsymbol{\Omega}\cdot\nabla)\mathbf{v}' - \boldsymbol{\Omega}(\nabla\cdot\mathbf{v}') \tag{2.6}$$

thus obtained can be considered to be the equation of motion of turbulent fluctuation arising in the vicinity of an arbitrary point P of the flow, which equation describes the instantaneous motion of a compressible barotropic fluid continuum in a relative coordinate system with the same velocity as the Reynolds mean velocity.

3. Stochastic turbulence model

The turbulent motion pattern arising in an arbitrary point of the flow field is examined in a suitably chosen special relative coordinate system that moves steadily at the same velocity as the Reynolds mean velocity in the given point and following von Kármán's [1] similarity hypothesis, it is assumed that, on the one hand, the turbulent motion patterns in all the points of the field are mechanically similar to each other, therefore all of them can be transformed into a common motion pattern, and, on the other hand, the viscosity of the medium filling the field of this common motion pattern is zero. Solving the equation of motion (2.6) of turbulent fluctuation in the relative coordinate system of the common motion pattern results in the scalar components of the velocity fluctuation, and they can be used to determine the scalar components of the Reynolds turbulent stress tensor, which, in line with the similarity hypothesis, are also components of the common motion pattern.

In flows surrounded by solid walls the velocity vector $\mathbf{v}$ and the vortex vector $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$ are not parallel vectors, and thus in these flows it is not always possible to interpret a special curvilinear orthogonal coordinate system, in which one set of coordinate lines is formed by the vortex lines (therefore one base vector is parallel to the vortex vector), i.e. in an arbitrary fixed point one coordinate surface will always be perpendicular to the vortex vector. If in this point the tangent plane of the other coordinate surface is given by the surface formed by the velocity vector $\mathbf{v}$ and the vortex vector $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$, then the normal of the third coordinate surface is necessarily parallel to the vector $\mathbf{v} \times (\nabla \times \mathbf{v})$, and the base vectors of the given special curvilinear orthogonal coordinate system (Figure 1) will be as follows:

$$\mathbf{e}'_3 = -\frac{\boldsymbol{\Omega}}{\Omega} \; ; \quad ; \quad \mathbf{e}'_2 = \frac{\mathbf{v} \times \boldsymbol{\Omega}}{|\mathbf{v} \times \boldsymbol{\Omega}|} = \frac{1}{\sqrt{1-S^2}} \left(\frac{\mathbf{v}}{v} \times \frac{\boldsymbol{\Omega}}{\Omega} \right)$$

$$\mathbf{e}'_1 = \mathbf{e}'_2 \times \mathbf{e}'_3 = \frac{1}{\sqrt{1-S^2}} \left(\frac{v}{v} - S\frac{\boldsymbol{\Omega}}{\Omega} \right) \quad ; \quad v = |\mathbf{v}| \quad ; \quad \Omega = |\boldsymbol{\Omega}| \quad ; \quad S = \frac{\mathbf{e}}{v} \cdot \frac{\boldsymbol{\Omega}}{\Omega} \; .$$

This special coordinate system will be the natural coordinate system in what follows. The transformation between the calculation coordinate system determined by the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and the previous natural coordinate system based on $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ can be performed by means of the tensor (and its transposed) given by the following dyadic product sums of the base vectors:

$$\boldsymbol{E} = \mathbf{e}'_1 \circ \mathbf{e}_1 + \mathbf{e}'_2 \circ \mathbf{e}_2 + \mathbf{e}'_3 \circ \mathbf{e}_3 \; ; \quad ; \quad \boldsymbol{E}^T = \mathbf{e}_1 \circ \mathbf{e}'_1 + \mathbf{e}_2 \circ \mathbf{e}'_2 + \mathbf{e}_3 \circ \mathbf{e}'_3 \; .$$

The transformation of vector and tensor fields from the natural coordinate system x'_1, x'_2, x'_3 into the calculation coordinate system x_1, x_2, x_3 is:

$$\mathbf{a}(x_1, x_2, x_3) = \boldsymbol{E} \cdot \mathbf{a}(x'_1, x'_2, x'_3) \tag{3.1}$$

$$\boldsymbol{A}(x_1, x_2, x_3) = \boldsymbol{E} \cdot \boldsymbol{A}(x'_1, x'_2, x'_3) \cdot \boldsymbol{E}^T \; . \tag{3.2}$$

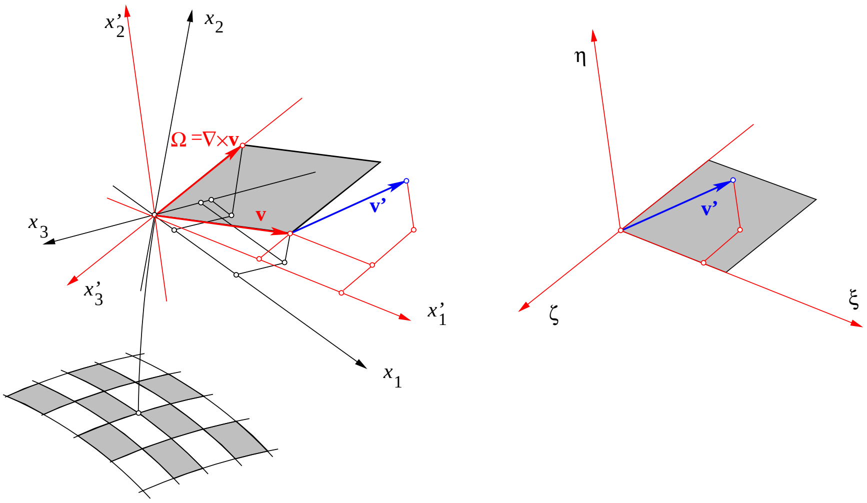


Figure 1. The three coordinate systems interpreted in an arbitrary point of the flow: the calculation coordinate system x_1, x_2, x_3, the natural coordinate system x'_1, x'_2, x'_3 and the relative coordinate system ξ, η, ζ with steady motion at a velocity equalling v.

Turbulent fluctuation is investigated separately from the turbulent main motion arising in the Reynolds mean-velocity field $\mathbf{v}(\mathbf{x}, t)$ in the relative coordinate system, which moves steadily at a velocity equalling the Reynolds mean velocity valid in the given point and whose axis is parallel to that of the natural coordinate system (Figure 1) and whose base vectors are the same as those of the natural coordinate system. In this relative coordinate system the differential operators ∇ and $\boldsymbol{\Omega} \cdot \nabla$ are as follows:

$$\nabla = \frac{\mathbf{e}'_1}{H'_1}\frac{\partial}{\partial x'_1} + \frac{\mathbf{e}'_2}{H'_2}\frac{\partial}{\partial x'_2} + \frac{\mathbf{e}'_3}{H'_3}\frac{\partial}{\partial x'_3} \quad ; \quad \boldsymbol{\Omega} \cdot \nabla = -\Omega(\mathbf{e}'_3 \cdot \nabla) = -\frac{\Omega}{H'_3}\frac{\partial}{\partial x'_3} \, ,$$

where H'_i ($i = 1, 2, 3$) means the Lamé metric coefficients of the relative coordinate system x'_1, x'_2, x'_3 , which are identical with the metric coefficients of the natural coordinate system. Next – also with respect to the relationship $\boldsymbol{\Omega'} = \nabla \times \mathbf{v}'$ – the equation of motion of turbulent fluctuation is obtained from equation (2.6) – assuming an incompressible medium – in the given relative coordinate system in the following form:

$$\frac{\partial \, (\nabla \times \mathbf{v}')}{\partial t} + (\mathbf{v}' \cdot \nabla)(\nabla \times \mathbf{v}') - [(\nabla \times \mathbf{v}') \cdot \nabla] \, \mathbf{v}' = -\frac{\Omega}{H'_3}\frac{\partial \mathbf{v}'}{\partial x'_3} \, . \tag{3.3}$$

According to hydro-mechanic evidence, we can talk about the similarity of two motion processes if their equations of motion can be transformed into each other by means of suitably chosen (geometric and dynamic) transformations. This requires that there exist accurately determined relationships between the scales (the conversion factors of the geometric and physical quantities in the equation of motion) used in the transformation, which will then create the necessary conditions of similarity. For the curvilinear orthogonal coordinates of place of the relative coordinate system

x'_1, x'_2, x'_3 the geometric transformations

$$H'_1 dx'_1 = M_L d\xi \quad ; \quad H'_2 dx'_2 = M_L d\eta \quad ; \quad H'_3 dx'_3 = M_L d\zeta \tag{3.4}$$

are introduced with the length scale M_L, which will transform the physical space of the velocity fluctuation into the points of the space described by the orthogonal coordinates ξ, η, ζ so that the point of origin $O(0,0,0)$ of the coordinate system ξ, η, ζ should correspond to the fixed point P (Figure 1). The corresponding base vectors of the two coordinate systems are identical with the base vectors of the natural coordinate system. Let us also introduce the physical transformations

$$dt = M_T d\tau \quad \text{and} \quad \mathbf{v}'\left(x'_1, x'_2, x'_3, t\right) = M_V \mathbf{w}'\left(\xi, \eta, \zeta, \tau\right) \tag{3.5}$$

for the time coordinate and the velocity fluctuation, respectively, where M_T is the time scale and M_V is the transformation scale of turbulent velocity fluctuation. This transformation maps the turbulent motion of the fluid continuum investigated arising in the vicinity of an arbitrary point P of the physical space onto the motion of a fluid model with zero viscosity filling an orthogonal coordinate system ξ, η, ς, and thus this coordinate system can be called the pattern space of the turbulence.

The turbulent motion patterns arising in the different points of the flow space can naturally only be considered to be mechanically similar to each other if from the equation of motion (3.3) of the turbulent fluctuation the given transformation can produce a differential equation for the dimensionless velocity fluctuation $\mathbf{w}'\left(\xi, \eta, \zeta, \tau\right)$, which does not depend on the characteristics of the motion in point P. After performing the transformation described in detail above, the differential equation (3.3) assumes the following form:

$$\frac{M_V}{M_T M_L} \frac{\partial \ \left(\nabla \times \mathbf{w}'\right)}{\partial \tau} + \frac{M_V^2}{M_L^2} \left[(\mathbf{w}' \cdot \nabla)(\nabla \times \mathbf{w}') - \left[(\nabla \times \mathbf{w}') \cdot \nabla\right] \mathbf{w}'\right] = -\frac{\Omega M_V}{M_L} \frac{\partial \mathbf{w}'}{\partial \zeta} \ ,$$

and this for the condition of mechanical similarity being searched for – i.e. the differential equation obtained for $\mathbf{w}'$ should not be dependent on the characteristics of motion in point P evidently results in the following equality between the 'constants' in the equation holding:

$$\frac{M_V}{M_T M_L} = \frac{M_V^2}{M_L^2} = \frac{\Omega M_V}{M_L} \ . \tag{3.6}$$

Namely now the former differential equation determining the vector function $\mathbf{w}'(\xi, \eta, \zeta, \tau)$ assumes the form

$$\frac{\partial \ \left(\nabla \times \mathbf{w}'\right)}{\partial \tau} + (\mathbf{w}' \cdot \nabla)(\nabla \times \mathbf{w}') - ((\nabla \times \mathbf{w}') \cdot \nabla)\, \mathbf{w}' = -\frac{\partial \mathbf{w}'}{\partial \zeta} \ , \tag{3.7}$$

which is in fact independent of the characteristics of motion in point P. However, according to the above, point P can be arbitrarily chosen, i.e. it may be any point in the turbulent flow space, consequently in every point of the flow space under examination the same dimensionless velocity fluctuation $\mathbf{w}'(\xi, \eta, \zeta, \tau)$ can be attached to the turbulent velocity fluctuation in the relative coordinate system ξ, η, ζ (the pattern space of the turbulence); and this at the same time expresses the mechanical similarity of turbulent velocity fluctuations and of the three-dimensional turbulence phenomenon itself as well.

The equality (3.6) between the conversion scales M_L, M_T and M_V contains two independent relationships. This means that one of the three scales can be chosen at will and that the development of the other two depends on this choice. Let e.g. the scale to be chosen at will be the length scale: $M_L = l$, then the others will take the forms, resp.:

$$M_T = 1/\Omega \quad ; \quad M_V = l\Omega \ . \tag{3.8}$$

The free choice of the length scale naturally only means here that its size l is indifferent in terms of the existence of mechanical similarity and that it is determined by other physical conditions.

In order to set up the turbulence model representing the internal mechanism of turbulent fluctuation, a particular solution of the differential equation (3.7) is to be found. The vector potential $\mathbf{\Psi}(\xi, \eta, \zeta, \tau)$ can be attached to the fluctuation velocity field, from which it is possible to derive the velocity field $\mathbf{w}'(\xi, \eta, \zeta, \tau)$ by forming the rotation: $\mathbf{w}' = \nabla \times \mathbf{\Psi}$. It is easy to understand that the vector $\nabla \times \mathbf{w}'$ can be written by using the three vectors $\mathbf{\Psi}$, $\mathbf{w}' = \nabla \times \mathbf{\Psi}$ and $\mathbf{u} = \mathbf{\Psi} \times (\nabla \times \mathbf{\Psi})$ as follows:

$$\nabla \times \mathbf{w}' = a\mathbf{\Psi} + b(\nabla \times \mathbf{\Psi}) + c\mathbf{\Psi} \times (\nabla \times \mathbf{\Psi}) \ ,$$

where a, b, c are scalar values. The vector $\nabla \times \mathbf{w}'$ thus broken down is substituted now into the differential equation (3.7), which gives the following:

$$a\left\{\frac{\partial \mathbf{\Psi}}{\partial \tau} + (\mathbf{w}' \cdot \nabla)\mathbf{\Psi} - (\mathbf{\Psi} \cdot \nabla)\mathbf{w}'\right\} + b\frac{\partial \mathbf{w}'}{\partial \tau} + c\left\{\frac{\partial \mathbf{u}}{\partial \tau} + (\mathbf{w}' \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{w}'\right\} = -\frac{\partial \mathbf{w}'}{\partial \zeta} \ .$$

However, the vector potential $\mathbf{\Psi}$ can also be chosen so that on the one hand $\nabla \cdot \mathbf{\Psi} = 0$, i.e. it is sourceless, and, on the other, its vector lines determined by the equation $\mathbf{\Psi} \times d\mathbf{x} = \mathbf{0}$ should remain unchanged in the viscosity-free fluid model filling the system ξ, η, ζ; and then in accordance with the Ertel-Friedmann conservation theorem (2.1), the two expressions in the figure bracket will disappear in the above equation. Finally, by introducing the term $\omega = -1/b$, the differential equation (3.7) to be solved assumes the following form:

$$\frac{\partial \mathbf{w}'}{\partial \tau} - \omega\frac{\partial \mathbf{w}'}{\partial \zeta} = 0 \ . \tag{3.9}$$

In order to solve this homogeneous first-order partial differential equation, the new independent variables $u = \zeta + \omega\tau$ and $z = \zeta$ are now introduced and then considering the identity $\mathbf{w}'(\xi, \eta, \zeta, \tau) \equiv \mathbf{w}'\left[\zeta(u, z), \tau(u, z); \xi, \eta\right]$ it is transformed for the coordinate planes u, z:

$$\frac{\partial \mathbf{w}'}{\partial z} = 0$$

The solution of this differential equation is the optional vector function of the form $\mathbf{w}'(u, \xi, \eta)$, which can be written by reversing the transformation in the following form:

$$\mathbf{w}'(\xi, \eta, \zeta, \tau) = \mathbf{w}'(\xi, \eta, \zeta + \omega\tau) \ .$$

Accordingly, the dimensionless velocity fluctuation $\mathbf{w}'$ can be described by an arbitrary vector function that has the following independent variables: ξ, η, and $\zeta + \omega\tau$.

Since the differential equation (3.9) to be solved is linear, its solution can also be given in the form of the following sum:

$$\mathbf{w}'(\xi,\eta,\zeta,\tau) = c\sum_{n=1}^{N}\begin{pmatrix} A_{1n}(\xi,\eta)\cos\left[n(\zeta+\omega\tau)+\alpha_{1n}\right] \\ A_{2n}(\xi,\eta)\cos\left[n(\zeta+\omega\tau)+\alpha_{2n}\right] \\ A_{3n}(\xi,\eta)\cos\left[n(\zeta+\omega\tau)+\alpha_{3n}\right] \end{pmatrix} - c\sum_{n=1}^{N}\begin{pmatrix} A_{2n}(\xi,\eta)\sin\left[n(\zeta+\omega\tau)+\alpha_{2n}\right] \\ A_{3n}(\xi,\eta)\sin\left[n(\zeta+\omega\tau)+\alpha_{3n}\right] \\ A_{1n}(\xi,\eta)\sin\left[n(\zeta+\omega\tau)+\alpha_{1n}\right] \end{pmatrix}, \tag{3.10}$$

where c, ω, and α_{in} $(i=1,2,3)$ are constants, and $A_{in}(\xi,\eta)$ $(i=1,2,3)$ are scalar functions, which, in line with the above, are to be chosen while satisfying the condition $\nabla\cdot\mathbf{w}'=0$. The phase angles $\alpha_{1n},\alpha_{2n},\alpha_{3n}$ are considered to be the three components of the three spatial dimensions, respectively, (and may be termed spatial phase angles), so the following relationship

$$\cos^2\alpha_{1n}+\cos^2\alpha_{2n}+\cos^2\alpha_{3n}=1 \tag{3.11}$$

has to hold between them. The physical meaning of the constant ω is: it is the smallest of the angular frequencies of the wave components of turbulent fluctuation. In order to determine the functions $A_{in}(\xi,\eta)$, substituting the vector function (3.10) into the equation of condition $\nabla\cdot\mathbf{w}'=0$ gives after a short calculation the following:

$$A_{1n}(\xi,\eta)=C_{1n}(\eta)e^{n\xi} \quad ; \quad A_{2n}(\xi,\eta)=C_{2n} \quad ; \quad A_{3n}(\xi,\eta)=C_{3n}(\xi)e^{-n\eta},$$

where C_{2n} is constant, and $C_{1n}(\eta)$ and $C_{3n}(\xi)$ are arbitrary functions (may also be constants). However, the origin $O(0,0,0)$ of the coordinate system ξ,η,ζ belongs to an arbitrary point of the physical space, where the dimensionless velocity vector of turbulent fluctuation can be given as

$$\mathbf{w}'(0,0,0,\tau)=c\mathbf{u}_0'(\tau),$$

where

$$\mathbf{u}_0'(\tau)=c\sum_{n=1}^{N}\begin{pmatrix} C_{1n}\cos(n\omega\tau+\alpha_{1n})-C_{2n}\sin(n\omega\tau+\alpha_{2n}) \\ C_{2n}\cos(n\omega\tau+\alpha_{2n})-C_{3n}\sin(n\omega\tau+\alpha_{3n}) \\ C_{3n}\cos(n\omega\tau+\alpha_{3n})-C_{1n}\sin(n\omega\tau+\alpha_{1n}) \end{pmatrix}. \tag{3.12}$$

When the dimensionless velocity fluctuation $\mathbf{w}'$ is known, the physical transformation (3.5) can be used to determine in the natural coordinate system x_1',x_2',x_3' the velocity fluctuation:

$$\mathbf{v}'(x_1',x_2',x_3',t)=M_V\mathbf{w}'(0,0,0,\tau)=cl\Omega\mathbf{u}_0'(\tau). \tag{3.13}$$

Since the vector function $\mathbf{u}_0'(\tau)$ is dedicated to describing the stochastic process of turbulent fluctuation, in the following the coefficients C_{in} will be considered to be random amplitudes and the values α_{in} to be random phase angles ($i=1,2,3$). One way of producing the coefficients C_{in} may be:

$$C_{in}=k_{in}\exp\left[-(n/K)^2\right],$$

where k_{in} are probability variables with even distribution in given intervals $[0,\delta_i]$, $|\delta_i|\leq 1$ and $K>>1$. The phase angles α_{in} $(i=1,2,3)$ are also probability variables with even distribution, among which, however, the relationship (3.11) holds, therefore only two of the three values can be chosen at will; so finally there appear five independent probability variables in each element of the function series (3.12). Therefore it can be stated that our stochastic turbulence model handles the scalar components of the dimensionless velocity fluctuation as the series sum of cosine waves with random phase and random amplitude.

When formula (3.13) of the turbulent velocity fluctuation $\mathbf{v}'$ is known, equation (1.10) can be used to determine the Reynolds turbulent stress tensor $\boldsymbol{F}_R$ in the natural coordinate system x_1', x_2', x_3':

$$\boldsymbol{F}_R(x_1', x_2', x_3', t) = -\rho(\overline{\mathbf{v}' \circ \mathbf{v}'}) = -\rho\ (l\Omega)^2 c^2 (\overline{\mathbf{u}_0' \circ \mathbf{u}_0'})\ .$$

If now the notations

$$\alpha = \overline{u_{0\xi}'^{\,2}} / \overline{u_{0\xi}' u_{0\eta}'} \quad ; \quad \beta = \overline{u_{0\eta}'^{\,2}} / \overline{u_{0\xi}' u_{0\eta}'} \quad ; \quad \gamma = \overline{u_{0\zeta}'^{\,2}} / \overline{u_{0\xi}' u_{0\eta}'}$$

$$\mu = \overline{u_{0\xi}' u_{0\zeta}'} / \overline{u_{0\xi}' u_{0\eta}'} \quad ; \quad \vartheta = \overline{u_{0\eta}' u_{0\zeta}'} / \overline{u_{0\xi}' u_{0\eta}'} \quad ; \quad \kappa^2 = -c^2 \overline{u_{0\xi}' u_{0\eta}'}$$

and the similarity tensor formed by using them

$$\boldsymbol{H} = \begin{pmatrix} \alpha & 1 & \mu \\ 1 & \beta & \vartheta \\ \mu & \vartheta & \gamma \end{pmatrix} \tag{3.14}$$

as well as the dominant turbulent shear stress

$$\Theta(x_1', x_2', x_3', t) = \rho\ (\kappa l \Omega\ )^2 \tag{3.15}$$

are introduced, then the former formula of the turbulent stress tensor $\boldsymbol{F}_R$ can be written in the following form:

$$\boldsymbol{F}_R(x_1', x_2', x_3', t) = \Theta(x_1', x_2', x_3', t) \boldsymbol{H}\ . \tag{3.16}$$

κ is a constant known in the literature as the Kármán-constant: $\kappa = 0.407$ (the free parameter c appearing in the formula (3.10) of the dimensionless velocity fluctuation serves to adjust it accurately). On the basis of the relationships (3.14) and (3.16) it can be realised that the physical meaning of the always positive Θ is: shear stress in the natural coordinate system.

Next, regarding turbulent temperature fluctuation also as the series sum of cosine waves with random phase and random amplitude, and similarly to the formulas (3.10)-(3.12) of the scalar components of dimensionless velocity fluctuation, the dimensionless temperature fluctuation

$$h'(\tau) = \hat{c} h_0'(\tau) = \hat{c} \sum_{n=1}^{N} \Big[\hat{C}_{1n} \cos(n\omega\tau + \hat{\alpha}_{1n}) - \hat{C}_{2n} \sin(n\omega\tau + \hat{\alpha}_{2n}) \Big] \tag{3.17}$$

is introduced where ω is identical with the value appearing in the formulas of dimensionless velocity fluctuation (thus the angular frequencies of the fluctuation will be identical in the velocity and temperature fields). The coefficients $\hat{C}_{1n}$, $\hat{C}_{2n}$ and the phase angles $\hat{\alpha}_{1n}$, $\hat{\alpha}_{2n}$ are probability variables with even distribution, and thus among the latter the relationship

$$\cos^2 \hat{\alpha}_{1n} + \cos^2 \hat{\alpha}_{2n} = 1 \tag{3.18}$$

holds (therefore the phase angles $\hat{\alpha}_{1n}$, $\hat{\alpha}_{2n}$ determine a planar direction). In a non-isotherm turbulent flow it may be assumed that the larger the inhomogeneity of the temperature field interpreted with the mean values is, the larger the turbulent temperature fluctuation may be; in other words, the temperature fluctuation T' is proportional with the absolute value of the gradient of the temperature field T interpreted in terms of mean values. On the basis of this, and similarly to the velocity fluctuation, the temperature fluctuation in the natural coordinate system x_1', x_2', x_3' can be written in the form

$$T'(x_1', x_2', x_3', t) = l(x_1', x_2', x_3')|\nabla T(x_1', x_2', x_3', t)|h'(\tau)$$

and thus the following

$$\overline{\mathbf{v}'T'} = l^2\Omega \ \ |\nabla T|\overline{w'(0,0,0,\tau)h'(\tau)} = l^2\Omega \ \ |\nabla T|c\hat{c}\overline{u_0'(\tau)h_0'(\tau)}$$

is obtained for the mean value according to time $\overline{\mathbf{v}'T'}$ – i.e. the velocity-temperature correlation – in the natural coordinate system. Now the notations

$$\kappa\hat{\kappa} = -c\hat{c}\overline{u_{0\eta}'h_0'} \quad ; \quad \delta = \overline{u_{0\xi}'h_0'}/\overline{u_{0\eta}'h_0'} \quad ; \quad \chi = \overline{u_{0\zeta}'h_0'}/\overline{u_{0\eta}'h_0'}$$

are introduced, where the free parameter $\hat{c}$ is used for the adjustment of the universal constant $\hat{\kappa} = 0.47$, the role of which in the turbulent boundary layers is similar to that of the Kármán-constant κ [3]. The direction vector

$$\hat{\mathbf{q}}(x_1', x_2', x_3') = \delta\mathbf{e'}_1 + \mathbf{e'}_2 + \chi\mathbf{e'}_3 \tag{3.19}$$

of the specific turbulent heat flux density is interpreted in the natural coordinate system x_1', x_2', x_3', and it can be used to write the velocity-temperature correlation $\overline{\mathbf{v}'T'}$ in the form

$$\overline{\mathbf{v}'T'} = -\kappa\hat{\kappa}l^2\Omega \ \ |\nabla T|\hat{\mathbf{q}}$$

and thus the vector of the specific turbulent heat flux density appearing in the equation of equilibrium of the internal energy (1.13) in the natural coordinate system x_1', x_2', x_3' is:

$$\mathbf{q}_{turb} = \rho c_P(\overline{\mathbf{v}'T'}) = -\rho c_P\kappa\hat{\kappa}l^2\Omega \ \ |\nabla T|\hat{\mathbf{q}} \ . \tag{3.20}$$

With the help of the formula of turbulent velocity fluctuation (3.13) the following can be determined in the natural coordinate system x_1', x_2', x_3' one after the other:

a) turbulent kinetic energy:

$$k = \frac{1}{2}\overline{\mathbf{v}' \cdot \mathbf{v}'} = -\frac{1}{2}(\alpha + \beta + \gamma)(\kappa l\Omega)^2 \ , \tag{3.21}$$

b) the vector of triple auto-correlation:

$$\overline{\mathbf{v}' \cdot \mathbf{v}' \cdot \mathbf{v}'} = (l\Omega)^3\hat{\mathbf{t}} \ , \tag{3.22}$$

$$\hat{\mathbf{t}} = \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} = c^3 \begin{pmatrix} \overline{u_{0\xi}'^{\,3}} + \overline{u_{0\xi}'u_{0\eta}'^{\,2}} + \overline{u_{0\xi}'u_{0\zeta}'^{\,2}} \\ \overline{u_{0\eta}'u_{0\xi}'^{\,2}} + \overline{u_{0\eta}'^{\,3}} + \overline{u_{0\eta}'u_{0\zeta}'^{\,2}} \\ \overline{u_{0\zeta}'u_{0\xi}'^{\,2}} + \overline{u_{0\zeta}'u_{0\eta}'^{\,2}} + \overline{u_{0\zeta}'^{\,3}} \end{pmatrix} \ , \tag{3.23}$$

c) turbulent dissipation:

$$\epsilon = \nu\overline{(\mathbf{v}' \circ \nabla) : (\mathbf{v}' \circ \nabla + \nabla \circ \mathbf{v}')} =$$

$$= -\nu\kappa^2 \left\{ \frac{2\alpha+\beta+\gamma}{{H'_1}^2}\left(\frac{\partial(l\Omega)}{\partial x'_1}\right)^2 + \frac{\alpha+2\beta+\gamma}{{H'_2}^2}\left(\frac{\partial(l\Omega)}{\partial x'_2}\right)^2 + \frac{\alpha+\beta+2\gamma}{{H'_3}^2}\left(\frac{\partial(l\Omega)}{\partial x'_3}\right)^2 + \right.$$

$$\left. +2\left(\frac{1}{H'_1H'_2}\frac{\partial(l\Omega)}{\partial x'q'_1}\frac{\partial(l\Omega)}{\partial x'_2} + \frac{\mu}{H'_1H'_3}\frac{\partial(l\Omega)}{\partial x'_1}\frac{\partial(l\Omega)}{\partial x'_3} + \frac{\vartheta}{H'_2H'_3}\frac{\partial(l\Omega)}{\partial x'_2}\frac{\partial(l\Omega)}{\partial x'_3}\right)\right\} . \tag{3.24}$$

Since the length scale $M_L = l$, time scale $M_T = 1/\Omega$ and velocity scale $M_V = l\Omega$ of turbulence obviously play a decisive role in the formulas of the stochastic turbulence model, the question arises: what is their relation with the length, time and velocity scales of Kolmogorov? The π-theory of dimension analysis will be of assistance in answering it. If we accept that the process of turbulent fluctuation (its internal mechanism) is fundamentally determined by the length scale l, the vortex intensity Ω, the specific turbulent dissipation ϵ and the kinematic viscosity ν of the medium, then these four parameters are the dominant physical characteristics of the turbulence, the dimensions of which expressed in terms of the basis dimensions – length L and time T in our case – assume the following forms:

target variable (a function of the rest):	ϵ turbulent dissipation	$\sim L^2/T^3$
geometrical variable:	l length scale	$\sim L$
process variable:	Ω vortex intensity	$\sim 1/T$
material variable:	ν kinematic viscosity	$\sim L^2/T.$

According to the π-theory of dimension analysis it follows from this that if the number of the dominant physical characteristics is: $n = 4$, and the number of basis dimensions is: $m = 2$, then the number of dimensionless physical characteristics determining the process is: $n - m = 2$; and between them exists an implicit relationship of the form $f(\pi_1, \pi_2) = 0$, where

$$\pi_1 = \frac{\nu}{l^2\Omega} \quad ; \quad \pi_2 = \frac{\epsilon}{l^2\Omega^3} \tag{3.25}$$

are the two dimensionless physical characteristics. Choosing the power function

$$f(\pi_1, \pi_2) = \pi_2 - C_E\pi_1^N = 0$$

as the implicit relationship, after substituting the dimensionless characteristics and ordering the equation, the relationship

$$\epsilon = C_E\nu^N(l\Omega)^{2(1-N)}\Omega^{N+1} \tag{3.26}$$

is given for the specific turbulent dissipation ϵ. From the analysis of experimental results the following conclusions can be drawn: on the one hand $N \approx 7/4$, and, on the other, the coefficient C_E is a constant directly proportional with the Reynolds number (calculated with the wall friction velocity).

It takes a short calculation to obtain from the formulas (3.25) the relationships

$$M_L = l = \left(\frac{\pi_2}{\pi_1^3}\right)^{1/4}\left(\frac{\nu^3}{\epsilon}\right)^{1/4} \quad ; \quad M_T = \left(\frac{\pi_2}{\pi_1}\right)^{1/2}\left(\frac{\nu}{\epsilon}\right)^{1/2} \quad ; \quad M_V = \frac{(\epsilon\nu)^{1/4}}{(\pi_1\pi_2)^{1/4}} \tag{3.27}$$

for the length scale l, the time scale $M_T = 1/\Omega$ and the velocity scale $M_V = l\Omega$ of the stochastic turbulence model. Since the length, time and velocity scales of Kolmogorov [4] are all determined by the formulas

$$M_{L,K} = \left(\nu^3/\epsilon\right)^{1/4} \quad ; \quad M_{T,K} = (\nu/\epsilon)^{1/2} \quad ; \quad M_{V,K} = (\epsilon\nu)^{1/4} \ ,$$

equations (3.27) express at the same time the relation between the scales of the stochastic turbulence model and the Kolmogorov scales.

It is worth mentioning that the dimensionless characteristics π_1 is nothing else than the reciprocal value of the Reynolds number $\mathrm{Re}_\Omega = l^2\Omega/\nu$ calculated with the peripheral velocity $l\Omega$ of the form (vortex) with radius l and angular velocity Ω . This Re_Ω-number is related to the turbulent Reynolds number, the Prandt $\mathrm{Re}_T = l\sqrt{k}/\nu$ as follows

$$\mathrm{Re}_T = \kappa\sqrt{-\frac{1}{2}(\alpha+\beta+\gamma)}\,\mathrm{Re}_\Omega \approx 0.767\,\mathrm{Re}_\Omega \ ,$$

thus the Re_Ω-number is a local dimensionless characteristic of the turbulence.

Summary. The stochastic turbulence model handles turbulent fluctuation processes as the series sum of cosine waves with random phase and random amplitude. It examines turbulent fluctuation, separated from the turbulent mainstream motion developing in the Reynolds mean velocity field, in a relative coordinate system moving steadily at a velocity identical with the Reynolds mean velocity; the base vectors of the coordinate system being identical with those of the natural coordinate system. In the relative coordinate system a particular solution of the equation of motion obtained for turbulent fluctuation can be used to produce one by one the following constant values: elements of the similarity tensor α, β, γ, μ, ϑ; scalar components $\hat{t}_1$, $\hat{t}_2$, $\hat{t}_3$ of the triple auto-correlation vector; components δ and χ of the direction vector of turbulent heat flux density. Series calculations were performed in order to determine the constants listed and they were compared with the results of several laboratory experimental examinations, and the following proposals were arrived at:

$$\alpha = -3.2 \quad ; \quad \beta = -1.6 \quad ; \quad \gamma = -2.4 \quad ; \quad \mu = \vartheta = 0$$

$$\hat{t}_1 = 1.3 \quad ; \quad \hat{t}_2 = -0.9 \quad ; \quad \hat{t}_3 = -1.0 \quad ; \quad \delta = -0{,}4 \quad ; \quad \chi = 0 \ .$$

These α, β, γ values also correspond to the measurement results published by Klebanoff in 1955 [5], according to which turbulence in the boundary layer along the planes is anisotropic, and the following proportionality

$$\overline{v_1'v_1'} : \overline{v_2'v_2'} : \overline{v_3'v_3'} = 4 : 2 : 3$$

holds between the normal components of the Reynolds turbulent stress tensor.

4. **Calculating turbulent flow based on the stochastic model**

In the Reynolds equation of motion of turbulent flow a divergence of the turbulent stress tensor $\boldsymbol{F}_R$ arises, which is the sum of the divergence of the deviator of the tensor and the gradient of its first scalar invariant. Thus the first scalar invariant of the turbulent stress tensor $\boldsymbol{F}_R$ achieves the same role in the equation of motion as the (thermodynamic) pressure p forming the first scalar invariant of the viscous stress tensor: in the equation of motion the negative gradients of both form part of the specific driving force. Going on with this train of thought: the deviator of the turbulent stress tensor $\boldsymbol{F}_R$ together with the friction stress tensor forming the deviator of the viscous stress tensor are responsible for the resistance of the fluid to deformation. Thus the apparent increase in viscosity inherent in the turbulent motion of the flowing fluid is caused by the deviator of the Reynolds turbulent stress tensor. Following from equation (3.16), the deviator $\boldsymbol{\sigma}_R$ of the stress tensor $\boldsymbol{F}_R$ can be expressed in the natural coordinate system x_1', x_2', x_3' with the deviator of the similarity tensor:

$$\boldsymbol{\sigma}_R(x_1', x_2', x_3', t) = \Theta(x_1', x_2', x_3', t)\boldsymbol{H}_* \ ,$$

where Θ is the dominant turbulent shear stress according to equation (3.15), and $\boldsymbol{H}_*$ is the deviator of the similarity tensor:

$$\boldsymbol{H}_* = \begin{pmatrix} \alpha_* & 1 & \mu \\ 1 & \beta_* & \vartheta \\ \mu & \vartheta & \gamma_* \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(2\alpha - \beta - \gamma) & 1 & \mu \\ 1 & \frac{1}{3}(2\beta - \gamma - \alpha) & \vartheta \\ \mu & \vartheta & \frac{1}{3}(2\gamma - \alpha - \beta) \end{pmatrix} .$$

Then the turbulent stress tensor $\boldsymbol{F}_R$ in the natural coordinate system can be written in the form

$$\boldsymbol{F}_R\,(x_1', x_2', x_3', t) = S_I\boldsymbol{I} + \Theta \;\; \boldsymbol{H}_* \ , \tag{4.1}$$

where $\boldsymbol{I}$ is the unit tensor, and S_I is the first scalar invariant of the stress tensor $\boldsymbol{F}_R$, which can be expressed using the specific turbulent kinetic energy k as follows:

$$S_I = -\frac{1}{3}\rho\left(\overline{\mathbf{v}_1'\mathbf{v}_1'} + \overline{\mathbf{v}_2'\mathbf{v}_2'} + \overline{\mathbf{v}_3'\mathbf{v}_3'}\right) = -\frac{2}{3}\rho k \ . \tag{4.2}$$

Since the natural coordinate system x_1', x_2', x_3' is not fixed in space, but changes from point to point in the flow space (Figure 1), it can only be used with limitations in the numerical solution of the concrete flow problems. It is expedient to perform the numerical calculations in a fixed coordinate system, consequently, the vector and tensor quantities have to be transformed from the natural coordinate system into a calculation coordinate system matching the problem to be solved according to equations (3.1)-(3.2). In the transformation the point-wise values of dominant turbulent shear stress Θ and scalar invariant S_I being scalar functions – do not change, and the transformed of the unit tensor $\boldsymbol{I}$ also remains a unit tensor, thus the transformation of the turbulent stress tensor $\boldsymbol{F}_R$ affects only the deviator $\boldsymbol{H}_*$:

$$\boldsymbol{G}(x_1, x_2, x_3, t) = \boldsymbol{E} \cdot \boldsymbol{H}_* \cdot \boldsymbol{E}^T \ . \tag{4.3}$$

Next the Reynolds turbulent stress tensor assumes the following form:

$$\boldsymbol{F}_R(x_1, x_2, x_3, t) = -\rho\frac{2}{3}k\boldsymbol{I} + \Theta\boldsymbol{G} \tag{4.4}$$

in the calculation coordinate system x_1, x_2, x_3. Since the apparent increase in viscosity mentioned is caused by the deviator $\boldsymbol{\sigma}_R = \Theta\boldsymbol{G}$ the deviator $\boldsymbol{G}$ defined by equation (4.3) will be called vortex viscosity tensor in what follows.

The tensor and its transformed of the natural x_1', x_2', x_3' and the calculation coordinate system x_1, x_2, x_3 can be written in the following matrix form:

$$\boldsymbol{E} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} \quad ; \quad \boldsymbol{E}^T = \begin{pmatrix} E_{11} & E_{21} & E_{31} \\ E_{12} & E_{22} & E_{32} \\ E_{13} & E_{23} & E_{33} \end{pmatrix} . \tag{4.5}$$

The scalar elements are determined by the scalar components (physical coordinates) of the velocity vector $\mathbf{v}$ and the vortex vector $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$ in the calculation coordinate system x_1, x_2, x_3:

$$E_{11} = \frac{1}{\sqrt{1-S^2}} \left(\frac{v_1}{v} - S\frac{\Omega_1}{\Omega} \right) \quad ; \quad E_{12} = \frac{1}{\sqrt{1-S^2}} \frac{v_2\Omega_3 - v_3\Omega_2}{v\Omega} \quad ; \quad E_{13} = -\frac{\Omega_1}{\Omega}$$

$$E_{21} = \frac{1}{\sqrt{1-S^2}} \left(\frac{v_2}{v} - S\frac{\Omega_2}{\Omega} \right) \quad ; \quad E_{22} = \frac{1}{\sqrt{1-S^2}} \frac{v_3\Omega_1 - v_1\Omega_3}{v\Omega} \quad ; \quad E_{23} = -\frac{\Omega_2}{\Omega}$$

$$E_{31} = \frac{1}{\sqrt{1-S^2}} \left(\frac{v_3}{v} - S\frac{\Omega_3}{\Omega} \right) \quad ; \quad E_{32} = \frac{1}{\sqrt{1-S^2}} \frac{v_1\Omega_2 - v_2\Omega_1}{v\Omega} \quad ; \quad E_{33} = -\frac{\Omega_3}{\Omega}$$

$$v = |\mathbf{v}| \quad ; \quad \Omega = |\nabla \times \mathbf{v}| \quad ; \quad S = \frac{v_1\Omega_1 + v_2\Omega_2 + v_3\Omega_3}{v\Omega} .$$

If now the expression (4.4) of the turbulent stress tensor $\boldsymbol{F}_R$ is substituted into the equation of motion (1.9), then after the introduction of the so-called total potential determined by the equation

$$\Pi = U + \frac{p}{\rho} - \frac{S_I}{\rho} = U + \frac{p}{\rho} + \frac{2}{3}k , \tag{4.6}$$

the Reynolds equation of motion describing the turbulent motion of the incompressible fluid is obtained in the following form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\Pi + \nu\Delta\mathbf{v} + \frac{1}{\rho}\mathrm{Div}(\Theta\boldsymbol{G}) . \tag{4.7}$$

This equation is the equation of equilibrium of the momentum transport of turbulent flow, where the terms have the following physical meanings: the two terms on the left-hand side are the substantial (complete in time) change of the momentum; the first term on the right-hand side is the specific driving force relating to mass unit, the second and the third terms are the specific resistance of the fluid to deformation, which result from, on the one hand, the molecular viscosity of the fluid, and, on the other, from the change of momentum resulting from the turbulent motion of the fluid particles.

In turbulent flow problems the equation of motion (4.7) and the equation of continuity

$$\nabla \cdot \mathbf{v} = 0 \tag{4.8}$$

expressing the conservation of mass of incompressible fluids are generally used as basic equations. But the Poisson-equation of the total potential Π may also be included among the basic equations, as it can be derived from the equation of motion (4.7) according to what follows under consideration of the equation of continuity (4.8). If the medium in flow is incompressible, then taking the divergence of each member in equation (4.7) gives the following equation:

$$\nabla \cdot (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\Delta\Pi + \frac{1}{\rho}\nabla \cdot \mathrm{Div}\,(\Theta \boldsymbol{G})$$

For an incompressible fluid – with respect to the expansion rule

$$(\mathbf{a} \circ \mathbf{b}){:}(\mathbf{c} \circ \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{b})$$

of the double scalar product of dyads – the following is obtained:

$$\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] = \nabla \cdot [(\mathbf{v} \circ \mathbf{v}) \cdot \nabla] = (\mathbf{v} \circ \nabla) : (\nabla \circ \mathbf{v}) \ ,$$

which gives the following Poisson-equation referring to the total potential:

$$\Delta\Pi = -(\mathbf{v} \circ \nabla) : (\nabla \circ \mathbf{v}) + \frac{1}{\rho}\nabla \cdot \mathrm{Div}\,(\Theta \boldsymbol{G}) \ . \tag{4.9}$$

It is easy to understand that supplementing the equation of motion (4.7) – either with the differential equation (4.8) or (4.9) – does not produce a closed system of equations in terms of solving the turbulent flow problem. Namely, in the system of equations formed by the four scalar differential equations the number of unknown functions – the three velocity components, the total potential Π and the dominant turbulent shear stress Θ – exceeds the number of equations, and thus the system of equations remains underdetermined. (The vortex viscosity tensor $\boldsymbol{G}$ does not increase the number of unknowns, for in its elements besides the constant elements of the similarity tensor $\boldsymbol{H}$ the velocity components and their place-derivates appear.)

Owing to its underdetermined character, in turbulent flow it is necessary to supplement this system of equations with further equations so that it can be solved. There are several possibilities for this.

a) For some turbulent flows with simpler geometrical configurations the scale function $l(\mathbf{x})$ of the turbulence is known, so it can be given in advance. In such a case it is possible to add the algebraic equation

$$\Theta(\mathbf{x}, t) = \rho\ (\kappa l \Omega)^2 \tag{4.10}$$

to the four partial differential equations and then the system of equations to be solved is formed by the four scalar partial differential equations and the algebraic equation (4.10) added. The system of equations formed by the total of five equations is suitable for determining the five unknowns: the three velocity components as well as Π and Θ; this is the *algebraic version* of the stochastic turbulence model.

b) The four partial differential equations are supplemented with a fifth differential equation, in which, however, new unknowns may appear, and thus further algebraic equations have to be added, in which there are no more new unknowns, which results in a closed system of equations formed by differential and algebraic equations. As the fifth differential equation it is possible to choose equation (3.24) of the stochastic turbulence model after it has been transformed into the calculation coordinate system

x_1, x_2, x_3:

$$\nu\kappa^2 \left\{(2\alpha+\beta+\gamma)\, A^2 + (\alpha+2\beta+\gamma)\, B^2 + (\alpha+\beta+2\gamma)\, C^2 + \right. \\ \left. +2\,(AB+\mu AC+\vartheta BC)\right\} + \epsilon = 0 \tag{4.11}$$

$$A = \frac{\partial(l\Omega)}{H'_1\partial x'_1} = E_{11}\frac{\partial(l\Omega)}{H_1\partial x_1} + E_{12}\frac{\partial(l\Omega)}{H_2\partial x_2} + E_{13}\frac{\partial(l\Omega)}{H_3\partial x_3}$$

$$B = \frac{\partial(l\Omega)}{H'_2\partial x'_2} = E_{21}\frac{\partial(l\Omega)}{H_1\partial x_1} + E_{22}\frac{\partial(l\Omega)}{H_2\partial x_2} + E_{23}\frac{\partial(l\Omega)}{H_3\partial x_3}$$

$$C = \frac{\partial(l\Omega)}{H'_3\partial x'_3} = E_{31}\frac{\partial(l\Omega)}{H_1\partial x_1} + E_{32}\frac{\partial(l\Omega)}{H_2\partial x_2} + E_{33}\frac{\partial(l\Omega)}{H_3\partial x_3}\ .$$

With this addition, however, the five unknowns – the three velocity components, Π and Θ – in the original four partial differential equations have been increased by two more: the velocity scale $l\Omega$ and the specific turbulent dissipation ϵ, thus another two algebraic equations have to be added in order to obtain a closed system of equations. One is (4.10), and the other is equation (3.26) of turbulent dissipation ϵ:

$$\epsilon = C_E \nu^N (l\Omega)^{2(1-N)} \Omega^{N+1}\ . \tag{4.12}$$

The system of equations to be solved – the original four + the supplementary (4.11) – is thus formed by five scalar partial differential equations, as well as the two algebraic equations (4.10) and (4.12) added; that is a total of seven equations. The number of unknowns is also seven: v_1, v_2, v_3, Π, Θ, $l\Omega$, and ϵ, thus the system of equations formed by the five partial differential equations + two algebraic equations is closed; this is the *one-equation version* of the stochastic turbulence model.

c) The equation of equilibrium (1.18) of turbulent kinetic energy can be used as the second differential equation to supplement the original four partial differential equations, after some formulas of the stochastic turbulence model have been introduced into it. The first such formula results from the formula (4.4) of the Reynolds turbulent stress tensor:

$$\overline{\mathbf{v}' \circ \mathbf{v}'} = \frac{2}{3}k\boldsymbol{I} - \frac{1}{\rho}\Theta\boldsymbol{G}\ , \tag{4.13}$$

the second is the formula of the velocity-pressure correlation $\overline{\mathbf{v}'p'}$, which, following Prandtl [6] is written in the form

$$\overline{\mathbf{v}'p'} = -\rho\nu_t \nabla k / C_k$$

where $C_k \approx 0.75$. Here ν_t is the Boussinesq vortex viscosity factor, which is equal to the product of the length scale l and the square root of the turbulent kinetic energy k; with respect to the relationship (3.21) of the turbulence model, the following is obtained for it:

$$\nu_t = l\sqrt{k} = \kappa a l^2 \Omega \quad ; \quad a = \sqrt{-\frac{1}{2}(\alpha+\beta+\gamma)}\ .$$

Now the formula of the velocity-pressure correlation assumes the following form:

$$\overline{\mathbf{v}'p'} = -\rho\frac{\kappa a}{C_k} l^2\Omega\ \ \nabla k\ . \tag{4.14}$$

The third formula results from the form of the triple auto-correlation determined by equation (3.22) in the natural coordinate system transformed into the calculation coordinate system:

$$\overline{\mathbf{v}' \cdot \mathbf{v}' \cdot \mathbf{v}'} = (l\Omega)^3 \mathbf{t} \quad ; \quad \mathbf{t} = \boldsymbol{E} \cdot \hat{\mathbf{t}} \ . \tag{4.15}$$

Finally, after substituting and ordering the relationships (4.13)-(4.15), the equation of equilibrium (1.18) of turbulent kinetic energy assumes the following form:

$$\frac{dk}{dt} = \frac{1}{\rho}\Theta\boldsymbol{G} : (\nabla \circ \mathbf{v}) - \epsilon - \nabla \cdot \left[\frac{1}{2}(l\Omega)^3\mathbf{t} - \left(\frac{5}{3}\nu + \frac{\kappa a}{C_k}\frac{(l\Omega)^2}{\Omega}\right)\nabla k + \frac{\nu}{\rho}\ \mathrm{Div}(\Theta\boldsymbol{G})\right] \ . \tag{4.16}$$

This can be used as the second supplementary equation of the four original partial differential equations given by the equation of continuity and the three Reynolds scalar equations of motion. Thus – supplementing the four original scalar differential equations with the partial differential equations (4.11) and (4.16) and adding the algebraic equations (4.10) and (4.12) – a closed system of equations with eight members is obtained for determining the total of eight unknown functions (v_1, v_2, v_3, Π, Θ, $l\Omega$, k and ϵ); this is the *two-equation version* of the stochastic turbulence model. It is easy to see that also when the latter version is chosen, the first step of the calculation is given by performing the one-equation version under b), and only then comes as a second step the solution of the differential equation (4.16) by using the functions determined above, which gives the distribution of turbulent kinetic energy $k(\mathbf{x}, t)$.

d) Finally it is worth noting that it is possible to choose the equation of equilibrium (4.16) as the fifth equation supplementing the four original partial differential equations instead of (4.11). Then due to the appearance of the three new unknown functions (k, ϵ and $l\Omega$), it becomes necessary to add three more algebraic equations: equations (4.10) and (4.12) and equation

$$k = (a\kappa l\Omega)^2 \tag{4.17}$$

resulting from (3.21) via the notation $a^2 = -(\alpha + \beta + \gamma)/2$; thus a closed system of equations is obtained, which may be called the *second one-equation version* of the stochastic turbulence model, and which corresponds – essentially – to the one-equation model published by Bradshaw, Ferris and Atwell [7]. Then it is no coincidence that the value $1/a^2 = 0.278$ in equation (4.17) – calculated with the constants of the stochastic turbulence model – is a good approximation of the Bradshaw-constant (≈ 0.3) [8] expressing the ratio between turbulent shear stress and turbulent kinetic energy.

In order to solve the underdetermined character of the differential equation system describing turbulent flow the first three of the four versions described above are evidently easier to handle than the fourth one, and according to our experience so far the results obtained from them give a good approximation of reality, versions a)-c) are to be given preference to version d).

In the knowledge of the distributions of the dominant turbulent shear stress $\Theta(\mathbf{x},t)$ and turbulent kinetic energy $k(\mathbf{x},t)$ the further scalar elements of the Reynolds turbulent stress tensor $\boldsymbol{F}_R$ can be determined on the basis of tensor equation (4.4) as:

$$\overline{v_i'v_i'} = \frac{2}{3}k - (\kappa l\Omega\ )^2 G_{ii} \quad ; \quad i = 1,2,3\ , \tag{4.18}$$

$$\overline{v_i'v_j'} = -(\kappa l\Omega)^2 G_{ij}\ . \tag{4.19}$$

It is worth noting here that these formulas of the scalar elements of the turbulent stress tensor can also be used in the one-equation version as well, if the distribution of turbulent kinetic energy k is calculated in the knowledge of the velocity scale $l\Omega$ using the relationship (4.17).

5. **Turbulent flow in a straight pipe with circular cross-section**

Let us take as an example a steady turbulent flow in a long straight pipe, which can be assumed to be rotation-symmetric and that only the axial component of the Reynolds time-mean average speed is not zero, and the other two components disappear. The computations are performed in the cylinder-coordinate system x,r,φ, and the following hold for the velocity and vortex components:

$$v_x = v(r) \quad ; \quad v_r = v_\varphi = 0 \quad ; \quad \Omega_x = \Omega_r = 0 \quad ; \quad \Omega_\varphi = -\frac{dv}{dr} \quad ; \quad \Omega = |\Omega_\varphi| = \left|\frac{dv}{dr}\right|\ .$$

From all this it also follows that the dominant turbulent shear stress is a function of only the coordinate r: $\Theta(r)$. The elements of the tensor $\boldsymbol{E}$ of the transformation between the natural and the calculation coordinate systems and of the vortex viscosity tensor $\boldsymbol{G}$ now take the forms:

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad ; \quad \boldsymbol{G} = \begin{pmatrix} \alpha_* & -1 & 0 \\ -1 & \beta_* & 0 \\ 0 & 0 & \gamma_* \end{pmatrix}\ .$$

A short calculation gives the following forms for the components in directions x and r of the divergence of the Reynolds turbulent stress tensor:

$$\mathrm{Div}\ (\Theta\boldsymbol{G})|_x = -\frac{1}{r}\frac{d(r\Theta)}{dr} \quad ; \quad \mathrm{Div}\ (\Theta\boldsymbol{G})|_r = \frac{\beta_*}{r}\frac{d(r\Theta)}{dr} - \frac{\gamma_*}{r}\Theta\ .$$

Then the two scalar equations of motion assume the forms:

$$0 = -\frac{\partial\Pi}{\partial x} + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) - \frac{1}{r}\frac{d(r\Theta)}{dr} \tag{5.1}$$

$$0 = -\frac{\partial\Pi}{\partial r} + \frac{\beta_*}{r}\frac{d(r\Theta)}{dr} - \frac{\gamma_*}{r}\Theta \tag{5.2}$$

and the first supplementary differential equation assumes the form:

$$\nu\kappa^2(\alpha + 2\beta + \gamma)\left(\frac{d(l\Omega)}{dr}\right)^2 + \epsilon = 0\ . \tag{5.3}$$

Since in these three differential equations there are altogether five unknown functions – velocity v, total potential Π, dominant turbulent shear stress Θ, velocity scale $l\Omega$

of fluctuation and specific turbulent dissipation ϵ –, two algebraic equations have to be added which contain no new unknown function:

$$\Theta = \rho(\kappa l\Omega)^2 \tag{5.4}$$

$$\epsilon = C_E \nu^N (l\Omega)^{2(1-N)} \left|dv/dr\right|^{N+1} \ . \tag{5.5}$$

It is easy to see that the system of equations (5.1)-(5.5) is a closed system from the point of determining the five unknown functions, thus this case represents an application of the *one-equation version* of the stochastic turbulence model.

In order to determine the elements of the turbulent stress tensor it is necessary to solve the second supplementary differential equation – the equation of equilibrium of the turbulent kinetic energy – as well:

$$\begin{aligned} 0 = {} & \frac{\Theta}{\rho}\frac{dv}{dr} + \varepsilon \\ & - \frac{1}{r}\frac{d}{dr}\left\{ r \left[\frac{\hat{t}}{2}\frac{dv}{dr}(l\Omega)^3 + \left(\frac{5\nu}{3} + \frac{\kappa a}{C_k}\frac{(l\Omega)^2}{\Omega}\right)\frac{dk}{dr} - \frac{\nu}{\rho}\left(\frac{\beta_*}{r}\frac{d(r\Theta)}{dr} - \frac{\gamma_*}{r}\Theta\right)\right]\right\} \ . \end{aligned} \tag{5.6}$$

In this differential equation only the specific turbulent kinetic energy k appears as a new unknown function, therefore the system of equations (5.1)-(5.6) forms a closed system regarding the determination of the six unknown functions: velocity v, total potential Π, dominant turbulent shear stress Θ, the velocity scale $l\Omega$ of turbulent fluctuation, turbulent dissipation ϵ, and turbulent kinetic energy k. The numerical solution of this system of equations with six members represents the *two-equation version* of the stochastic turbulence model.

Equations (5.1)-(5.2) make it easy to see that the partial derivative of the total potential $\Pi(x, r)$ according to x is constant, which can be expressed by the drop in potential $\Delta\Pi$ measured on the pipe-section of length L as follows: $\partial\Pi/\partial x = -\Delta\Pi/L$. As usual, the viscous shear stress on the pipe wall is used to define the wall friction velocity v_*:

$$v_* = \sqrt{\frac{|\tau_{wall}|}{\rho}} = \sqrt{\frac{\Delta\Pi R}{2L}} \ ,$$

where R is the pipe radius. If the Reynolds number $\mathrm{Re}_* = v_* R/\nu$ calculated with the wall friction velocity v_* as well as the dimensionless place coordinate $\xi = r/R$, the dimensionless velocity $V = v/v_*$ and the dimensionless velocity scale $Y = l\Omega/v_*$ are introduced, the differential equations (5.1) and (5.3) after the substitution of the algebraic equations (5.4) and (5.5) assume the forms:

$$\frac{dV}{d\xi} = \mathrm{Re}_* \left(\kappa^2 Y^2 - \xi\right) \tag{5.7}$$

$$a_* \kappa \frac{dY}{d\xi} = \frac{\sqrt{C_E}}{\mathrm{Re}_*^{(N-1)/2}} Y^{1-N} \left|\frac{dV}{d\xi}\right|^{\frac{N+1}{2}} \ , \tag{5.8}$$

where on the basis of analysing the measurement results $N \approx 7/4$, and

$$a_* = \sqrt{-\alpha - 2\beta - \gamma} \ .$$

The coefficient C_E depends, beyond depending on the Re_*-number, on the spatial coordinate ξ:

$$C_E = C_0\,(1-\xi)^3 \quad ; \quad C_0 = (0.09\dots0.16)\,\mathrm{Re}_* \ .$$

Reducing equations (5.7) and (5.8) gives for the velocity scale Y the differential equation

$$\frac{dY}{d\xi} = \frac{\mathrm{Re}_*\sqrt{C_E}}{a_*\kappa}Y^{1-N}\left|\kappa^2Y^2-\xi\right|^{\frac{N+1}{2}} \ .$$

It is easy to see that the introduction of the function

$$U = Y^N$$

results in the differential equation to be solved assuming the form

$$\frac{dU}{d\xi} = N\frac{\mathrm{Re}_*\sqrt{C_E}}{a_*\kappa}\left|\kappa^2U^{2/N}-\xi\right|^{\frac{N+1}{2}} \ ,$$

which is easy to integrate numerically with the Runge-Kutta method. When the distribution $Y(\xi)$ is known, integration of the equation (5.7) determines the dimensionless velocity distribution $V(\xi)$. The dimensionless velocity maximum playing the role of integration constant depends on the Re_*-number:

$$V_{\max} = 6 + \frac{1}{\kappa}\ln\mathrm{Re}_* \ . \tag{5.9}$$

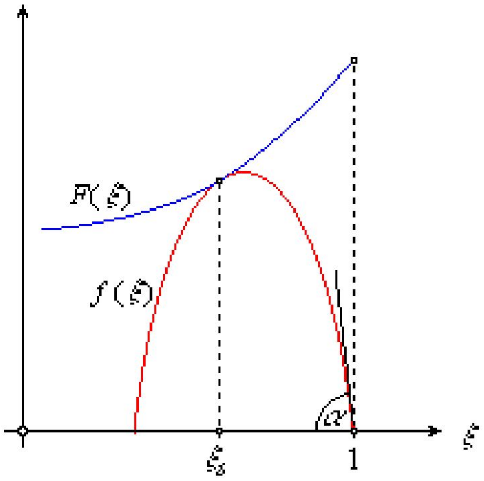


Figure 2. Smooth fitting of the turbulent distribution onto distribution in a viscous sub-layer

With respect to the fact that the differential equation (3.9) forming the basic equation of the stochastic turbulence model and the vector function (3.10) obtained as its solution for turbulent velocity fluctuation are valid for the fully developed turbulent motion – and this restriction applies to all the relationships derived from it –, the differential equations (5.7)-(5.8) do not hold in the thin viscous sublayer adhering to the pipe wall. As a result, in the calculation of the functions $Y(\xi)$ and $V(\xi)$, the zero boundary conditions relating to the pipe wall can be fulfilled by the smooth fitting of the viscous sublayer onto the distributions disappearing next to the wall. For the turbulent distribution $F(\xi)$, the polynomial of the form

$$f(\xi) = (a + b\xi)\,(1 - \xi) \quad ; \quad \xi_\delta \le \xi \le 1 \tag{5.10}$$

is used to approximate the smoothly fitting distribution in place ξ_δ, where the coefficients a and b can be determined by fulfilling the conditions of smooth fitting, according to which the two functions fitting each other and their first derivatives are identical in place ξ_δ of the fitting (Figure 2), which results in the following:

$$a = F(\xi_\delta)\frac{1 - 2\xi_\delta}{(1-\xi_\delta)^2} - F'(\xi_\delta)\frac{\xi_\delta}{1-\xi_\delta} \quad ; \quad b = F(\xi_\delta)\frac{1}{(1-\xi_\delta)^2} + F'(\xi_\delta)\frac{1}{1-\xi_\delta} \; .$$

Experimental experience shows that the derivative of the velocity distribution $V(\xi)$ on the wall (in place $\xi = 1$) both for the laminar and turbulent cases is

$$V'(1) = -\,\mathrm{Re}_* \; .$$

For $f(\xi)$ replacing $F(\xi)$

$$f'(1) = -\tan\alpha = -(a + b)$$

holds (Figure 2), thus for $V(\xi)$the condition

$$a + b = \mathrm{Re}_*$$

has to hold, which, after the substitution of the previous formulas of the coefficients a and b gives the equation

$$2V(\xi_\delta) + (1 - \xi_\delta)\,[v'(\xi_\delta) - \mathrm{Re}_*] = 0 \tag{5.11}$$

for determining the coordinates ξ_δ of smooth fitting. When ξ_δ is known, it is possible to calculate also the coordinates a and b of the distributions in the viscous sublayer not only in terms of the dimensionless velocity scale $Y(\xi)$ and the dimensionless velocity $V(\xi)$, but also in terms of the other turbulent distribution as well. It follows from all this that the place coordinate ξ_δ determined by means of equation (5.11) can be considered to be the boundary of the viscous sublayer, where the turbulent distributions smoothly fit the laminar distributions fulfilling the zero boundary condition valid on the pipe wall.

When the dimensionless velocity scale $Y(\xi)$ is known, the as yet unknown further turbulent distributions can be determined: the dimensionless distribution of the dominant turbulent shear stress of the form

$$\frac{\Theta}{\rho v_*^2} = \kappa^2 Y^2 \tag{5.12}$$

and the dimensionless distribution of the specific turbulent dissipation of the form

$$\frac{\epsilon R}{v_*^3} = C_E\,\mathrm{Re}_*\,Y^{2(1-N)}\left|\kappa^2 Y^2 - \xi\right|^{N+1} \; . \tag{5.13}$$

Finally the integration of differential equation (5.2) gives for the dimensionless total potential the equation

$$\frac{\Pi(x,\xi)}{v_*^2} = \frac{\Pi_0}{v_*^2} - \frac{\Delta\Pi}{v_*^2}\frac{x}{L} + \beta_*\kappa^2 Y^2 + (\beta_* - \gamma_*)\kappa^2 \int\limits_0^\xi Y^2 \frac{d\xi}{\xi} \tag{5.14}$$

for the integration constant Π_0/v_*^2 neglecting the field forces, the following form is obtained:

$$\frac{\Pi_0}{v_*^2} = \frac{p_0}{\rho v_*^2} - \kappa^2 \frac{\alpha+\beta+\gamma}{3} \ ,$$

where p_0 is pressure in the middle of the initial cross-section of the pipe section (in place $x = 0$) and $\Delta\Pi$ is the total potential drop in section with length L. The calculation of the distribution of the total potential gives the last step of the *one-equation version* of the stochastic turbulence model.

In order to determine the distribution of the specific turbulent kinetic energy it is necessary to solve the – second supplementary – differential equation (5.6) numerically. When the distribution $Y(\xi)$ is known, the differential equation (5.6) can be made with some modification – including one integration in terms of ξ – suitable for determining the distribution of the dimensionless specific turbulent kinetic energy $K = k/v_*^2$:

$$\left\{\frac{5}{3} + \frac{\kappa a}{C_k}\frac{Y^2}{|\kappa^2Y^2 - \xi|}\right\}\frac{dK}{d\xi} - 2\kappa\frac{\beta_*}{a_*}\,\mathrm{Re}_*\,\sqrt{C_E}Y^{2-N}\left|\kappa^2Y^2 - \xi\right|^{\frac{N+1}{2}} +$$

$$+\,\mathrm{Re}_*\,\frac{\hat{t}_2}{2}Y^3 - \kappa^2(\beta_* - \gamma_*)\frac{Y^2}{\xi} - \mathrm{Re}_*^2\,F(\xi) = 0 \ , \tag{5.15}$$

where $C_k \approx 0.75$, and

$$F(\xi) = \frac{1}{\xi}\int\limits_0^\xi \left\{\kappa^2Y^2\left(\kappa^2Y^2 - \xi\right) + C_E Y^{2(1-N)}\left|\kappa^2Y^2 - \xi\right|^{N+1}\right\}\xi d\xi \ .$$

When the distribution of the dimensionless velocity scale $Y(\xi)$ is known (i.e. it has been determined on the basis of the one-equation version), there is no obstacle in the way of the numerical solution of the differential equation (5.15). For the integration constant K_0 the formula

$$K_0 = -\kappa^2\frac{\alpha+\beta+\gamma}{2}$$

can be recommended, which is also supported by measurement results.

Next – when the distributions $K(\xi)$ and $Y(\xi)$ are known – the formulas (4.18)-(4.19) can be used to calculate the scalar elements different from zero of the Reynolds turbulent stress tensor as well:

$$F_{R_{11}} = \frac{\overline{v_x'v_x'}}{v_*^2} = 2K(\xi)\frac{\alpha}{\alpha+\beta+\gamma} \quad ; \quad F_{R_{22}} = \frac{\overline{v_r'v_r'}}{v_*^2} = 2K(\xi)\frac{\beta}{\alpha+\beta+\gamma}$$

$$F_{R_{33}} = \frac{\overline{v_\varphi'v_\varphi'}}{v_*^2} = 2K(\xi)\frac{\gamma}{\alpha+\beta+\gamma} \quad ; \quad F_{R_{12}} = \frac{\overline{v_x'v_r'}}{v_*^2} = \kappa^2Y^2(\xi) \ .$$

In the viscous sublayer – similarly to equations (5.7)-(5.8) and for the same reason – the differential equation (5.15) also loses its validity, therefore the zero boundary conditions referring to the pipe wall for the main stresses can also be fulfilled by smooth fitting on the distributions disappearing on the wall. The connection between the distributions in turbulent and sublayer zones can be approximated by a third order polynomial for $\overline{v'_x v'_x}/v_*^2$ and by a second polynomial for the other two main stresses. The place ξ_F of the fitting of the velocity distribution $V(\xi)$ is in the vicinity of the place of fitting of ξ_δ: $\xi_F \leq \xi_\delta$ (it is smaller for larger Re_*-numbers). In a second order approximation the main stress distribution $F(\xi)$ in the viscous sublayer is replaced by the following function

$$f(\xi) = F(\xi_F)\frac{(1-\xi)(b+\xi)}{(1-\xi_F)(b+\xi_F)} \quad ; \quad \xi_F \leq \xi \leq 1 \, ,$$

where the coefficient b can be determined as follows:

$$b = \frac{1-(B+2)\xi_F}{B+1} \quad ; \quad B = \frac{F'(\xi_F)}{F(\xi_F)}(1-\xi_F) \; .$$

In a third-order approximation the main stress distribution $F(\xi)$ in the viscous sublayer is replaced by the function

$$f(\xi) = F(\xi_F)\frac{(1-\xi)(\xi^2+b\xi+c)}{(1-\xi_F)(\xi_F^2+b\xi_F+c)} \quad ; \quad \xi_F \leq \xi \leq 1 \, .$$

In this case the value of the maximum of the main stress – assumed in the viscous sublayer – is also to be given, which according to experimental experience is a function of the Re_*-number:

$$f_m = \frac{7\,\mathrm{Re}_* - 475}{\mathrm{Re}_* - 50}$$

and then the coefficients b and c as well as the place of maximum ξ_m are to be determined iteratively on the basis of the formulas

$$b = \frac{(B+1)(2-3\xi_m)\xi_m + (B+3)\xi_F^2 - 2\xi_F}{(B+1)(2\xi_m-1)-(B+2)\xi_F+1}$$

$$c = (2-3\xi_m)\xi_m + (1-2\xi_m)b \, ,$$

while the condition $f(\xi_m) = f_m$ is simultaneously fulfilled. Determining also the distribution of the scalar elements of the Reynolds turbulent stress tensor in the viscous sublayer has performed the last step in the *two-equation version* of the stochastic turbulence model.

Figure 3 shows the solutions of the differential equations (5.7), (5.8) and (5.15) describing the turbulent flow in a pipe with circular cross-section for three different Re_*-numbers ($\mathrm{Re}_{*_1} = 100$, $\mathrm{Re}_{*_2} = 1000$ and $\mathrm{Re}_{*_3} = 10000$) against the dimensionless place coordinate $y^+ = \mathrm{Re}_*(1-r/R)$. Notations: $V(y^+)$ is velocity, $Y(y^+)$ is velocity scale and $K(y^+)$ is turbulent kinetic energy (all are made dimensionless by means of the wall friction velocity v_*). For the sake of better comparison, the Figure also includes the Prandtl universal turbulent velocity profile $V = \frac{1}{\kappa}\ln y^+ + 5$ and the laminar velocity distribution $V = y^+$ of the viscous sublayer. Our experience shows that for the stability of the calculation it is necessary to choose the division of the interval $0 \leq \xi \leq 1$ to be at least 5000; and so that it increases with the Re_*-number (5000 is sufficient for $\mathrm{Re}_* = 100$, while 15000 is recommended for $\mathrm{Re}_* = 10000$).

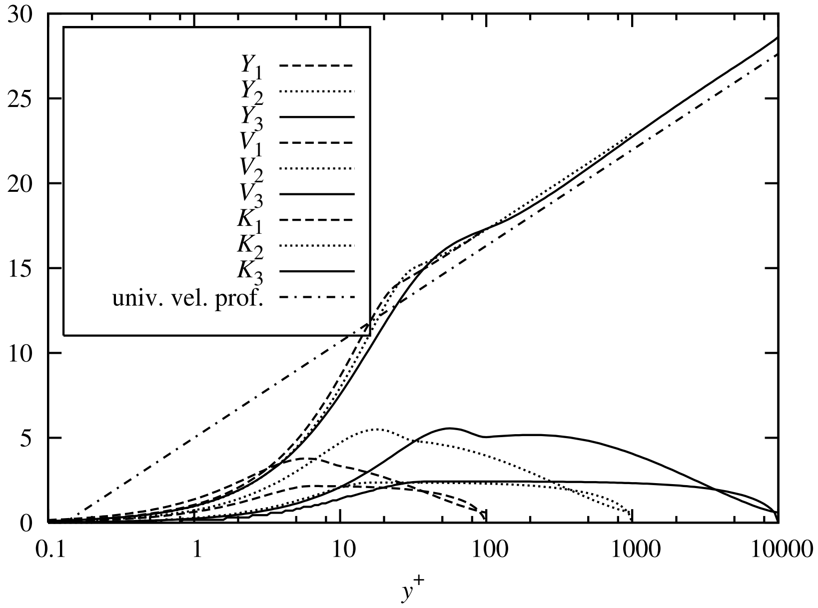


Figure 3. Solutions for Y, V, K of the differential equation of turbulent flow in a pipe for three different Re_*-numbers.

Figure 4 shows a comparison of calculated velocity distribution for two different Reynolds-numbers with the measurement results of Laufer [9]; the Reynolds-numbers of the identification are calculated using the velocity maximum because that was used in the measurements as well. Since it is expedient to use the number $\mathrm{Re}_* = v_* R/\nu$ calculated with the wall friction velocity v_* for identifying the numerical calculations, the Re_m and Re_* numbers: $\mathrm{Re}_{m_a} = 50000$ and $\mathrm{Re}_{*_a} = 1079.44$ as well as $\mathrm{Re}_{m_b} = 500000$ and $\mathrm{Re}_{*_b} = 8826.69$ corresponding to each other in the two cases shown in the Figure are given. The figures show a good correlation between the calculated and measured results.

Figure 5 shows a comparison between the calculated results of the scalar elements along the main diagonal of the Reynolds turbulent stress tensor (i.e. the main stresses) while Figure 6 shows a comparison between the calculated results of turbulent shear stresses as well as those of velocity profiles in terms of the velocity maximum and the measurement results of Laufer [9] for the same two Re_m-numbers.

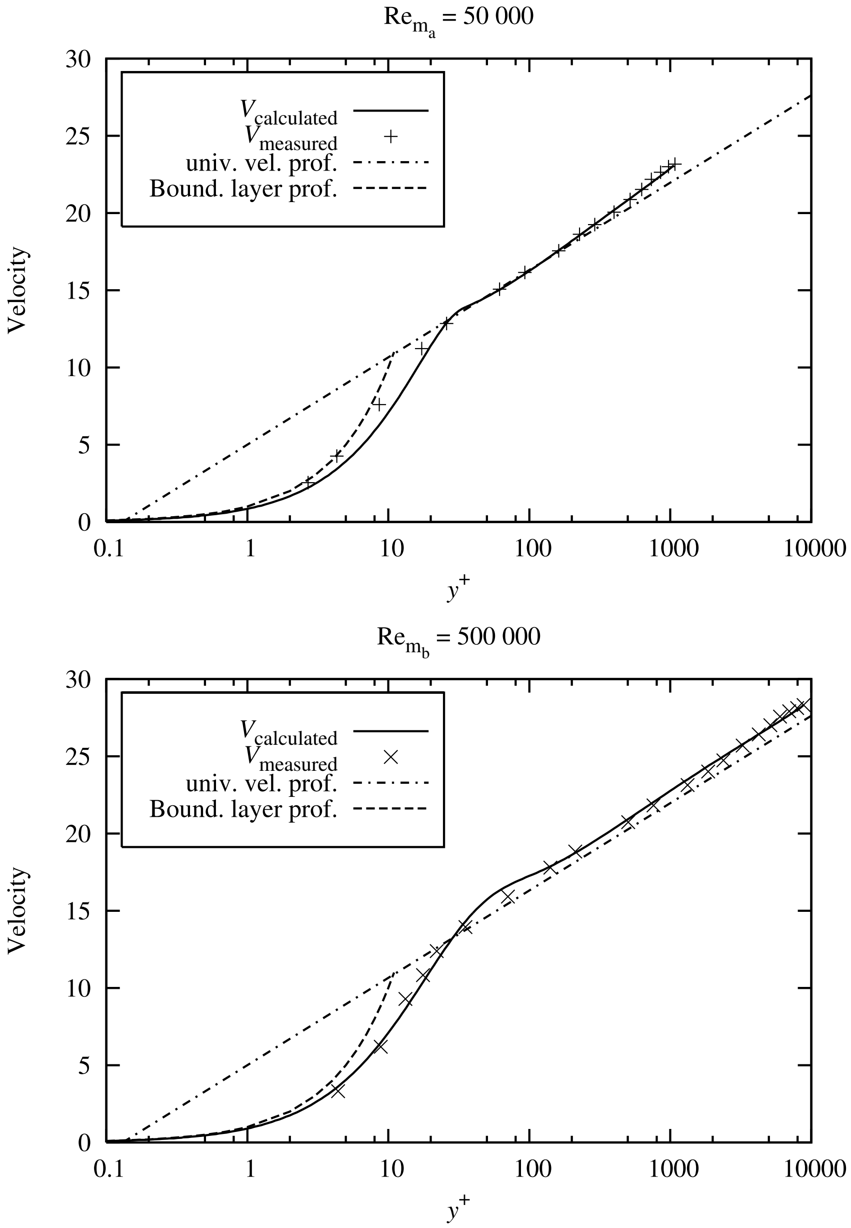
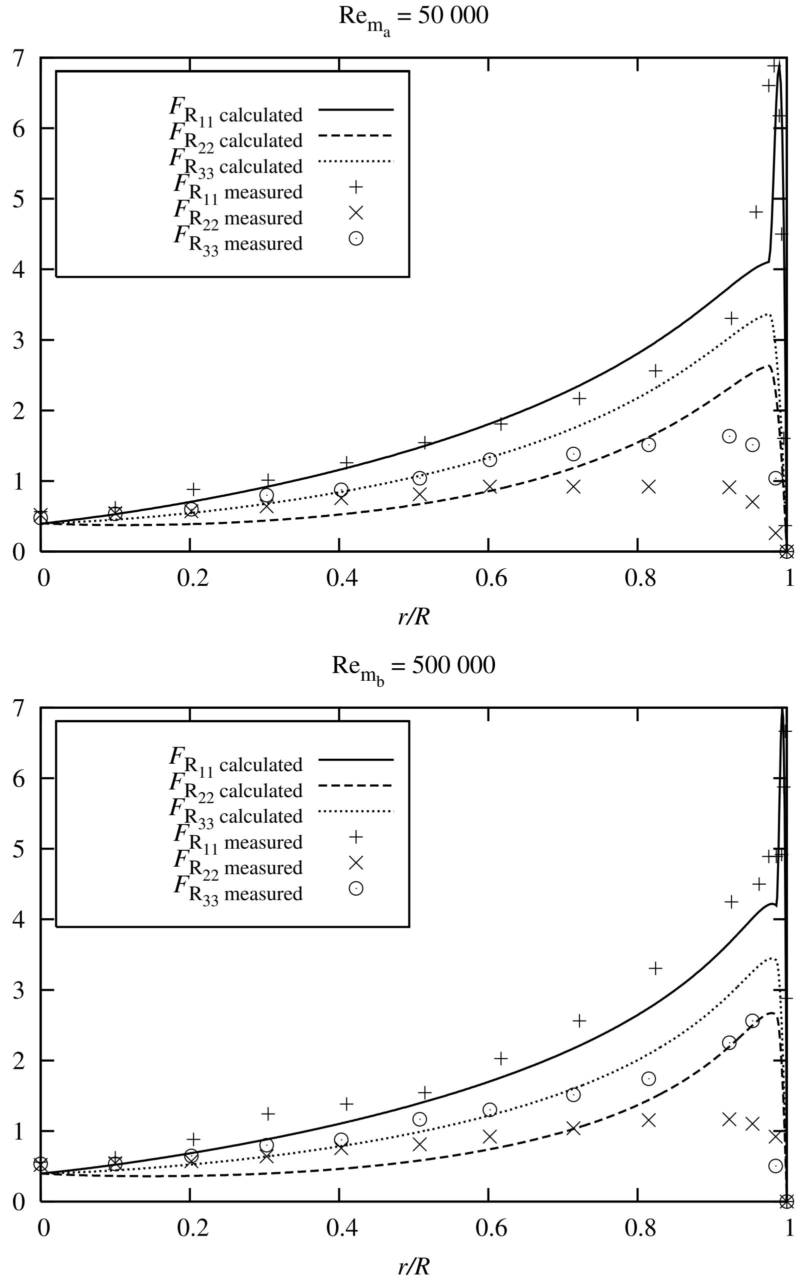


Figure 4. Comparison of calculated velocity distributions with the measurement results of Laufer [9].


$\mathrm{Re}_{m_a} = 50\,000$

$\mathrm{Re}_{m_b} = 500\,000$

Figure 5. Comparison of calculated turbulent main stresses for two different Re-numbers with the measurement results of Laufer [9].

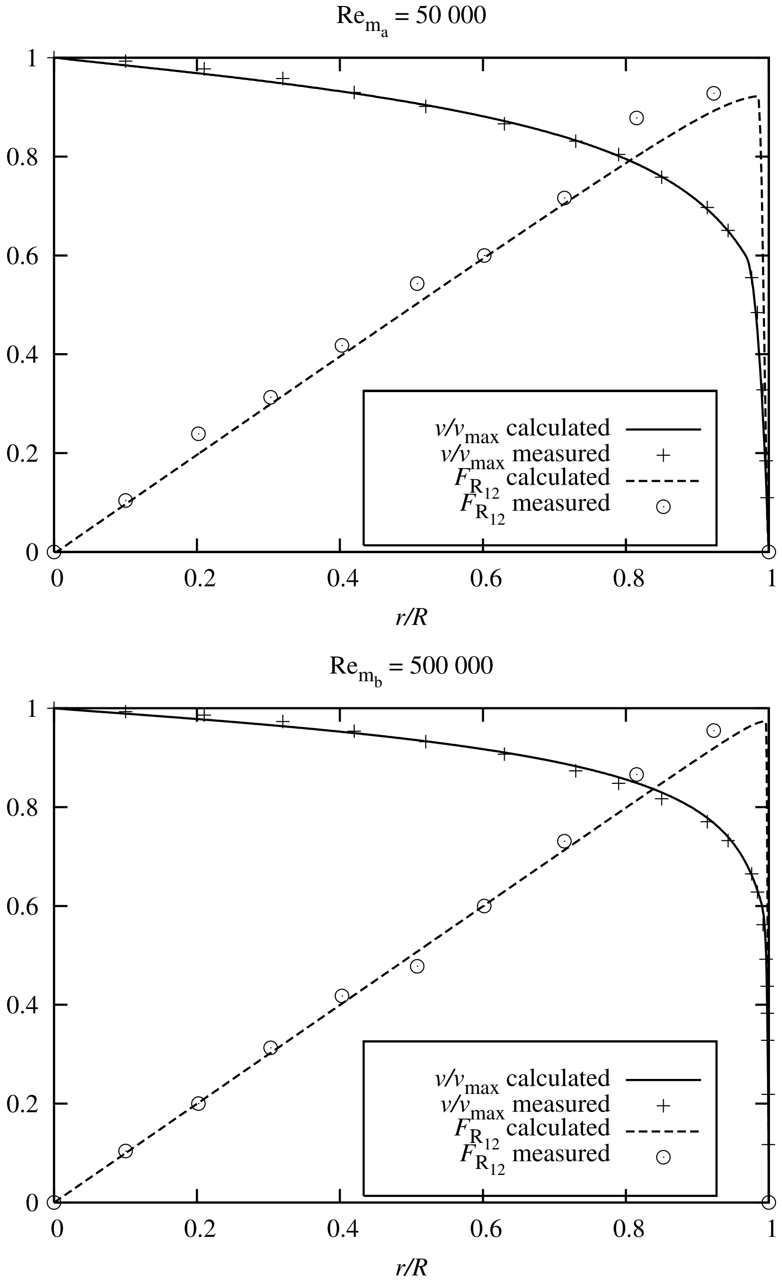


Figure 6. Comparison of the calculated results for the velocity profile and turbulent shear stress with the measurement results of Laufer [9].

For the sake of completeness let us sum up in short the formulas forming the *algebraic version* of the stochastic turbulence model as well. If the expression

$$Y = \frac{l\Omega}{v_*} = \frac{l}{R}\left|\frac{dV}{d\xi}\right| \tag{5.16}$$

is substituted into differential equation (5.7), and the equation – of second order for $dV/d\xi$ – obtained is solved, a differential equation of the form

$$\frac{dV}{d\xi} = \frac{1 \pm \sqrt{1 + 4\kappa^2 \operatorname{Re}_*^2 (l/R)^2 \xi}}{2\kappa^2 \operatorname{Re}_* (l/R)^2}$$

is obtained, where evidently the lower one of the double sign has physical reality (i.e. for a flow in pipes in the points of the cross-section $dV/d\xi \leq 0$). Taking this into account, the above equation is modified to some extent, then its integration produces an integral expression for determining the dimensionless velocity distribution $V(\xi)$:

$$V(\xi) = V_{\max} - \int_0^{\xi} \frac{2 \operatorname{Re}_* \xi \;\; d\xi}{1 + \sqrt{1 + 4\kappa^2 \operatorname{Re}_*^2 (l/R)^2 \xi}} \,, \tag{5.17}$$

where the integration constant $V_{\max}$ is to be calculated using relationship (5.9). For determining the dimensionless velocity scale, on the basis of (5.16) the following formula is obtained:

$$Y(\xi) = \frac{2 \operatorname{Re}_* (l/R) \xi}{1 + \sqrt{1 + 4\kappa^2 \operatorname{Re}_*^2 (l/R)^2 \xi}} \,. \tag{5.18}$$

By using the following dimensionless scale function, calculation results with a good correspondence with the laboratory measurements are obtained:

$$l/R = \left(0.35 + 0.15\xi^2\right)\left(1 - \xi^2\right)\left\{1 - \exp\left[m(\xi - 1)\right]\right\} \,,$$

where $m = \operatorname{Re}_* /25$. When the distributions $Y(\xi)$ and $V(\xi)$ are known – as it has already been expounded concerning the one-equation version –, the formulas (5.12)-(5.14) can be used to determine the following one after the other: the dimensionless dominant turbulent shear stress $\Theta/\rho v_*^2$, the dimensionless specific turbulent dissipation $\epsilon R/v_*^3$ and the dimensionless total potential Π/v_*^2.

Finally it should be noted that by using a special scale function the integration can be performed in the formula (5.17), resulting in a closed analytic expression for the turbulent profile fully developed in the pipe. The dimensionless scale function is

$$l/R = \left(1 - \xi^3\right)/3 \,, \tag{5.19}$$

which substituted into equation (5.17) results after integration in the formula:

$$V(\xi) = V_m - \frac{1}{\kappa} \ln \frac{1 + \xi^{3/2}}{1 - \xi^{3/2}} \,. \tag{5.20}$$

Since this distribution does not automatically satisfy the zero boundary condition referring to the pipe wall either (i.e. it has a pole in place $\xi = 1$), the zero boundary condition can be fulfilled in this case in the viscous sublayer by a smoothly fitted distribution (Figure 2).

It should be noted that the turbulent velocity distribution calculated by formulas (5.17) and (5.20) of the algebraic version of the stochastic turbulence model is identical 'within line thickness' with the velocity distribution calculated on the basis of the one- and two-equation versions, which in turn shows a very good correspondence with the measurement results.

6. **Summary**

The underdetermined character of the system of differential equations formed by the equation of continuity and the three Reynolds scalar equations of motion can be eliminated on the basis of the stochastic turbulence model by adding the differential equation for the turbulent velocity scale, and the turbulent mean velocity field can be determined by solving the system of equations with a suitable numerical method. When the velocity scale is known, the distribution of the turbulent shear stress can be calculated. This method – where the original system of differential equations containing four members is supplemented with a further differential equation – is termed the *one-equation* version of the stochastic turbulence model.

In order to be able to determine the scalar elements along the main diagonal of the Reynolds turbulent stress tensor, the system of equations to be solved is to be supplemented with a second differential equation as well, i.e. the transport equation of the specific turbulent kinetic energy. After determining the distribution of the turbulent kinetic energy it becomes possible to calculate the distributions of the turbulent main stresses by using the formulas of the stochastic turbulence model. The method obtained – in a three dimensional case built on numerically solving a system of equations constituted by a total of six differential equations and two algebraic equations – forms the *two-equation version* of the stochastic turbulence model.

A comparison of the calculations performed for a flow in a pipe with a circular cross-section for two different Reynolds-numbers with the experimental results of Laufer [9] shows that the results calculated on the basis of the stochastic turbulence model exhibit a satisfactory correspondence with the measurements; for the distribution of velocity and shear stress the calculated and measured results are practically identical; for the turbulent main stresses the correspondence at present can be interpreted with a certain limit of error, which can be improved by a correction of the constants in the model.

REFERENCES

1. Kármán, Th. v.: Mechanische Ähnlichkeit und Turbulenz *Proc. 3rd Congress of Applied Mechanics* (1930), 86–93.
2. Kotschin, N. J., Kiebel, L. A. and Rose, N. W.: *Theoretische Hydromechanik* Akademie Verlag Berlin (1954) Vol. I, 138–142.
3. Gersten, K. and Herwig, H.: *Strömungsmechanik* Vieweg Braunschweig/Wiesbaden (1992), p 465.
4. Kolmogorov, A. N.: The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Numbers *Dokl. Akad. Nauk. SSSR 30*, 1941, pp 301–305.

5. KLEBANOFF, P. S.: *Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient* NACA Report 1247, 1955.
6. PRANDTL, L.: Über ein neues Formelsystem für die ausgebildete Turbulenz *Nachr. Akad. Wiss. Göttingen, Math. Phys.* Klasse 1945, 6–9.
7. BRADSHAW, P., FERRIS, D.H. and ATWELL N.P.: Calculation of Boundary Layer Development Using the Turbulent Energy Equation *Journal of Fluid Mechanics*, **28**, Pt. 3, (1967), 593–616.
8. WILCOX, D. C.: *Turbulence Modeling for CFD (2nd. ed.)* DCW Industries 2000, pp 40, 279.
9. LAUFER, J.: *The Structure of Turbulence in Fully Developed Pipe Flow* NACA Report 1174, 1954.