A new procedure of reduction to pole of magnetic data – improved noise rejection capability

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Summary

In geophysical research it is an important objective to find more accurate measuring and more effective data processing methods. Measurement data always contain noise, which can mislead the interpreter, or hide useful information. The often used traditional DFT algorithm shows low noise rejection capability. On the other hand there are robust methods to solve the overdetermined inverse problem with excellent noise rejection capabilities. Therefore we suggest a new inversion based Fourier transformation method, where the continuous frequency spectrum is discretized with series expansion and the series expansion coefficients as model parameters are determined in the framework of the Iteratively Reweighted Least Squares (IRLS) algorithm using the so-called Cauchy-Steiner weights. In this paper the method was tested on noisy synthetic magnetic data generated above two magnetic bodies. The results prove the successful applicability of our inversion based S-IRLS-FT algorithm.

Introduction

In geophysical research one endeavors to do more accurate measurements, more effective data processing and particular interpretation. Measurement data always contain noise, which can mislead the interpreter, or hide useful information. Therefore the noise reduction capability of data processing method is an important characteristic. The Fourier-transform (FT) or rather its discrete version (DFT) is an often used tool for signal processing, which provides connection between the time domain of signal registration and the frequency domain of signal processing. Since the measured data always contain noise and it is linearly transformed into the frequency domain, DFT is very noise sensitive. On the other hand there are a collection of robust methods to solve the overdetermined inverse problem with excellent noise rejection capabilities. Therefore □ proposed to handle the one dimensional Fourier transform of noisy data as an overdetermined inverse problem. The essence of the method is that the continuous frequency spectrum is discretized according to the series expansion and the series expansion coefficients as model parameters are determined in the framework of the Iteratively Reweighted Least Squares (IRLS) algorithm using the so-called Cauchy-Steiner weights. In this paper, the generalized two dimensional case is presented and the new method is applied in the reduction to pole of the synthetic magnetic data. In the numeric experiment, the very good noise rejection capability is proved.

The overview of the inversion based 1D Fourier transform algorithm

Using the terminology of discrete inverse problem theory, the direct problem giving the theoretical values of the time domain data is given by the inverse Fourier-transform

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\omega) \cdot e^{j\omega t} d\omega \,,$$

where *U(ω)* denotes the frequency spectrum, which is a complex and continuous function. To define the Fourier transform as an inverse problem, first the discretization have to be made. Therefore we assume that the Fourier spectrum is given in the form of series expansion

$$U(\omega) = \sum_{n=1}^{M} B_n \cdot \Psi_n(\omega) \tag{1}$$

with the B_n complex valued expansion coefficients and $\Psi_n(\omega)$ basis functions. With Eq. (1) the calculated signal in the k -th sampling time is

$$u^{(theor)}(t_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sum_{n=1}^{M} B_n \cdot \Psi_n(\omega) \right) \cdot e^{j\omega t_k} d\omega = \sum_{n=1}^{M} B_n G_{k,n} \,, \tag{2}$$

where

$$G_{k,n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi_n(\omega)\, e^{j\omega t_k} d\omega = \mathcal{F}^{-1}\{\Psi_n(\omega)\} \tag{3}$$

is an element of the $N \times M$ Jacobian matrix (N is the number of time domain data and M is the number of unknown series expansion coefficients). It can be seen that the Jacobian is the inverse Fourier transform of the $\Psi_n(\omega)$ basis function. Fulfilling all of the requirements for the basis functions the set of the scaled Hermite functions were chosen which consist of the Hermite polynomials generated by means of the Rodriguez formula

$$h_n(\omega,\alpha) = (-1)^n e^{\alpha\omega^2} \left(\frac{d}{d\omega} \right)^n e^{-\alpha\omega^2} \tag{4}$$

and are calculated as

$$H_n(\omega,\alpha) = \frac{e^{-\frac{\alpha\omega^2}{2}} \cdot h_n(\omega,\alpha)}{\sqrt{\sqrt{\frac{\pi}{\alpha}}\, n!(2\alpha)^n}} \;. \tag{5}$$

$H_n^{(0)}(\omega)$ of Hermite functions (with $\alpha = 1$) have the special property that they are the eigenfunctions of the Fourier-transform

$$\mathcal{F}\left\{H_n^{(0)}(t)\right\}=(-j)^n H_n^{(0)}(\omega) \ , \tag{6}$$

and for the inverse Fourier transform

$$\mathcal{F}^{-1}\left\{H_n^{(0)}(\omega)\right\}=(j)^n H_n^{(0)}(t) . \tag{7}$$

$H_n(\omega,\alpha)$ as

$$G_{k,n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(\omega,\alpha) \cdot e^{j\omega t}\, d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt[4]{\alpha}\, H_n^{(0)}(\omega) \cdot e^{j\omega t}\, d\omega . \tag{8}$$

Considering the notation $\omega t = \omega' t'$ and $\omega' = \sqrt{\alpha}\,\omega,\ t' = \frac{t}{\sqrt{\alpha}}$ we get

$$G_{kn} = \frac{1}{\sqrt[4]{\alpha}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n^{(0)}(\omega')\ e^{j\omega' t'}\, d\omega' = \frac{1}{\sqrt[4]{\alpha}}\ \mathcal{F}^{-1}\left\{H_n^{(0)}(\omega')\right\}, \tag{9}$$

which is the inverse Fourier transform of the basis function. According to Eq. (7) the Jacobian matrix can be rewritten as

$$G_{k,n} = \frac{1}{\sqrt[4]{\alpha}} (j)^n H_n^{(0)}(t') . \tag{10}$$

Thus the Jacobian matrix can be calculated without integration, which makes a practical improvement to the inversion based Fourier transform method. In order to make the Fourier transform more robust the Iteratively Reweighted Least Squares (IRLS) method using Cauchy-Steiner weights was implemented. In order to achieve the optimal values of the unknown parameters (*B_n*) the following weighted norm is minimized

$$E_w = \sum_{k=1}^{N} w_k e_k^2 \ ,$$

where the weighting matrix is defined in the Most Frequent Value method of Steiner (1997). The deviation vector given in the above formula is $e_k = u_k^{measured} - u_k^{theor}$.

The weighted norm gives reliable results for inverse problems even if the measured data set contains outliers. The obtained normal equation can be written in the j-th iteration step

$$\mathbf{G}^T \mathbf{W}^{(j-1)} \mathbf{G}\, \vec{B}^{(j)} = \mathbf{G}^T \mathbf{W}^{(j-1)} \vec{u}^{\,measured} \ .$$

This iterative process continues until a proper stop criterion is met and the series expansion coefficients are determined. The one dimension Fourier spectrum can be calculated at any frequency by using Eq. (1). The above algorithm was tested on noisy data set containing outliers and an

□ 2014).

The generalization for 2D

By generalizing the introduced method for two dimensions, the two dimensional inverse Fourier-transform means the forward problem and provides the theoretical values of the space domain

$$f(x,y) = \iint_{(-\infty,+\infty)} F(u,v) e^{j2\pi(ux+vy)} du dv \ , \tag{11}$$

where $F(u,v)$ stands for the 2D space-frequency spectrum, which will be discretized using the scaled Hermite functions defined above

$$F(u,v) = \sum_{n=1}^{N} \sum_{m=1}^{M} B_{n,m} H_n(\alpha,u) H_m(\beta,v) . \tag{12}$$

Using Eq. (11) the data calculated at the point (x_k, y_l)

$$f(x_k,y_l) = \sum_{n=1}^{N} \sum_{m=1}^{M} B_{n,m} \int_{-\infty}^{+\infty} H_n(\alpha,u) e^{j2\pi u x_k} du \int_{-\infty}^{+\infty} H_m(\beta,v) e^{j2\pi v y_l} dv \ , \tag{13}$$

where $k = (1,2,...,K)$, $l = (1,2,...,L)$ denote the sequence number of the measurement points along the x and y directions, respectively. Introducing the Jacobian matrix according to Eq. (8) we can write

$$f(x_k,y_l) = f_{k,l} = \sum_{n=1}^{N} \sum_{m=1}^{M} B_{n,m} G_{k,l}^{n,m} . \tag{14}$$

Following the calculations shown in Eqs. (8)-(9) and using the properties of the basic Hermite function in Eqs. (6)-(7) we find the Jacobian as

$$G_{k,l}^{n,m}=\frac{1}{\sqrt[4]{\alpha\beta}}\,\mathbf{F}^{-1}\left\{H_n^{(0)}(u')\right\}\mathbf{F}^{-1}\left\{H_m^{(0)}(v')\right\}=\frac{1}{\sqrt[4]{\alpha\beta}}(j)^{(n+m)}H_n^{(0)}(x_k')H_m^{(0)}(y_l'),$$

where $u'=\sqrt{\alpha}u, v'=\sqrt{\beta}v$, $x_k'=\frac{x_k}{\sqrt{\alpha}}, x_k'=\frac{y_l}{\sqrt{\beta}}$.

Let us transform the indices $i=n+(m-1)N,\quad s=k+(l-1)K$ to make the programming of the algorithm easier. With this notation the total number of the unknown expansion coefficient is $I=N*M$ and that for the measurement data is $S=K*L$ thus Eq. (14) gives the form $f_{k,l}=f_s=\sum_{i=1}^{I}B_iG_{s,i}$ with $i=(1,2,...,I),\ s=(1,2,...,S)$. After this the deviation of the measured and calculated data is given as $e_s=f_s^{measured}-\sum_{i=1}^{I}B_iG_{s,i}$ and the weighted inverse problem can be defined by minimizing the functional $E_w=\sum_{s=1}^{N}w_se_s^2$ with the w_s Cauchy-Steiner weights used in the one dimensional IRLS algorithm.

Investigation on noisy magnetic data

In order to test the robust 2D S-IRLS-FT method a synthetic magnetic data set was generated. Two magnetic bodies were placed onto the measurement area, which has an extension of (-120, 120) m in both of x and y direc magnetization of 200 nT has a magnetization of 150 nT (with D=2.5° declination and I=63° inclination). A rectangular measurement system was assumed with 5 m spacing in both directions. In order to simulate noisy data set the magnetic data were contaminated by random noise following Cauchy distribution. The sizes and locations of magnetic bodies and the surface magnetic data calculated by Kunarathnam (Kunaratnam, 1981) are shown in Figure 1.

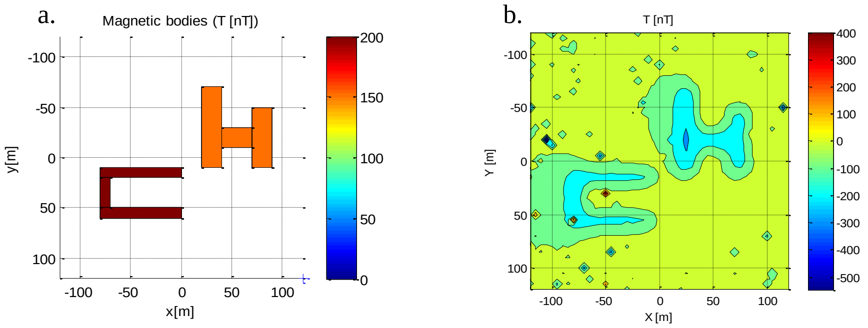

Figure 1 *a.) Magnetic bodies of the synthetic measurement area, b.) Noisy magnetic data before pole reduction.*

The interpretation of magnetic measurements are often supported by reducing the measurements to the I=90° pole. This can be done in the space frequency domain by applying the formula

$$R(u,v)=T(u,v)S(u,v), \tag{15}$$

where *T(u,v)* is the 2D Fourier transform of the magnetic data set, *S(u,v)* is the frequency domain operator of the pole reduction. The reduced data set in space domain can be determined by inverse Fourier transformation of the *R(u,v)* data set.

Figure 2a represents the magnetic map reduced to the pole using the conventional DFT algorithm. Peaks with high magnetization can be seen around the magnetic bodies. These are the residuals of added noise remained after the 2D DFT showing its low noise reduction capability. Figure 2b shows the calculated spectrum and the high noise is clearly visible. In contrary the results obtained by the new S-IRLS-FT algorithm (Figure 3a) show smooth magnetic field around the objects, proving the high noise suppression capability of the method. It is also confirmed by the calculated spectrum plotted in Figure 3b.

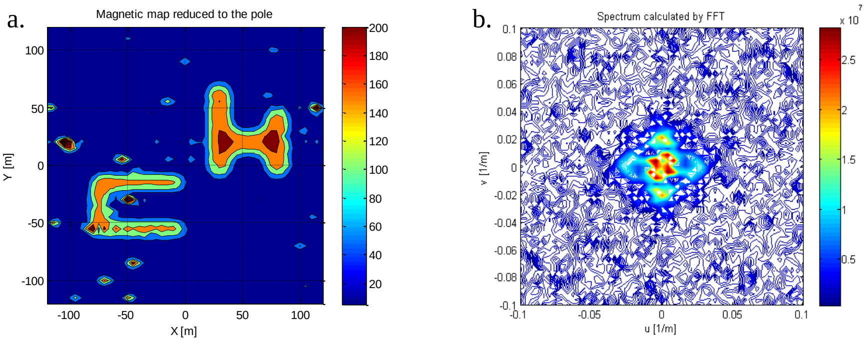

Figure 2 *a.) The noisy surface magnetic data after pole reduction using 2D DFT, b.) The spectrum of the noisy magnetic data.*

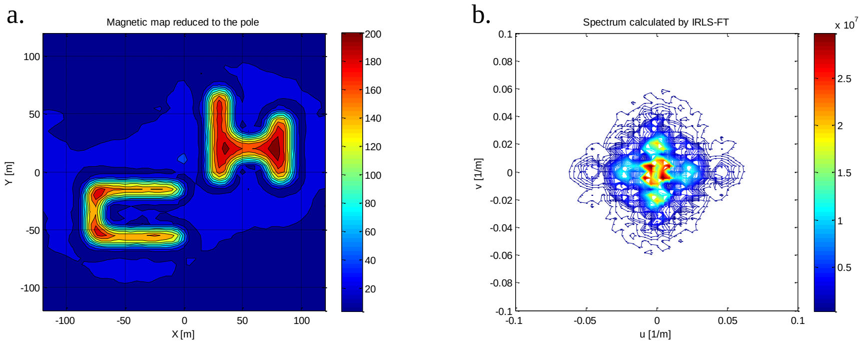

Figure 3 *a.) The noisy surface magnetic data after pole reduction using the new 2D IRLS-FT algorithm, b.) The 2D IRLS-FT of the noisy magnetic data.*

Discussion and conclusions

A new inversion based 2D Fourier transform algorithm was introduced to enhance the noise suppression during the reduction to pole of magnetic data. To avoid the calculation of complex integrals and advantage that the Hermite functions are eigen-functions of the Fourier transform was taken. The new 2D S-IRLS-FT method using Cauchy-Steiner weights was numerically tested by using synthetic magnetic data. The noise reduction capability of the new inversion-based Fourier transform method was proved. Comparing to the conventional DFT, significant improvement was shown by applying in the pole-reduction of magnetic data.

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