STATIC ANALYSIS OF TWO-LAYERED PIEZOELECTRIC BEAMS WITH IMPERFECT SHEAR CONNECTION

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Abstract

The object of the present paper is to analyse the static behaviour of the piezoelectric two-layered beams with interlayer slip. The Euler-Bernoulli hypothesis is assumed to hold for each layer and a linear constitutive equation between the horizontal slip and the interlaminar shear force is considered. For each beam component the applied electric field is constant and the prescribed mechanical loads are not present.

1. INTRODUCTION

Piezoelectric beams are commonly used in many engineering applications, such as control elements and transducers. Piezoelectric beams in unimorph and bimorph configurations, in particular, are widely used for systems where actuation and/or sensing are needed [1,2]. The present paper deals with the static analysis of two-layered piezoelectric beams with interlayer slip. The connection of the beam components in axial direction is weak but in normal direction is perfect. There is no separation in normal direction between the beam components. The interlayer slip in axial direcition is defined on the common boundary of beam components as the difference of the axial components of the displacement field. The considered two-layered piezoelectric beam configuration is shown in Fig. 1. The cross-section of beam component B_i is a rectangle A_i of which sides are b, h_i $(i=1,2)$. The centre of the cross-section A_i is C_i $(i=1,2)$ and the Y-weighted centre of the whole cross-section is C as shown in Fig. 1. The Young modulus of beam component B_i is denoted by Y_i $(i=1,2)$. From Fig. 1. it follows that

$$c_1 = \left|\overrightarrow{CC_1}\right| = \frac{A_2Y_2}{\langle AY\rangle}c, \quad c_2 = \left|\overrightarrow{CC_2}\right| = \frac{A_1Y_1}{\langle AY\rangle}c, \quad c = c_1 + c_2, \tag{1}$$

$$\langle AY\rangle = A_1Y_1 + A_2Y_2, \quad A_1 = h_1b, \quad A_2 = h_2b. \tag{2}$$

According to the Euler-Bernoulli beam theory the displacements are as follows [3]

$$u = u_i(x) - z\frac{\mathrm{d}w}{\mathrm{d}x}, \quad v = 0, \quad w = w(x), \quad (x,y,z)\in B_i, \quad (i=1,2), \tag{3}$$

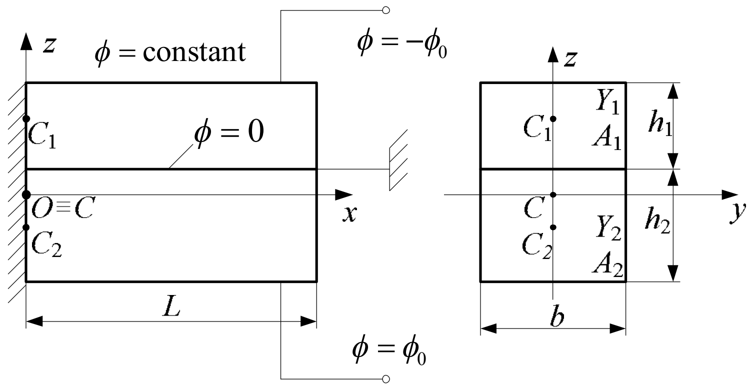

Figure 1. Two-layered piezoelectric beam with weak shear connection.

where u, v and w are the displacements in x, y and z directions. The next constitutive equation will be used

$$\sigma_x = Y\varepsilon_x - e_{31}E_z. \tag{4}$$

Here, σ_x is the normal stress, ε_x is the normal strain, E_z is the z component of the electric field vector and e_{31} is the piezoelectric constant of the thickness polarized beam component. E_z in terms of potential ϕ is obtained as (Fig. 1.)

$$E_{z1} = \frac{\phi_0}{h_1}, \quad E_{z2} = \frac{\phi_0}{h_2} \tag{5}$$

in beam components B_1 and B_2. The top and bottom surfaces of piezoelectric layers are metalized. Next we use $e_{31i} = e_i \quad (i = 1,2)$. By the use of Eqs. (4), (5) the expression of the normal stress σ_x can be given as

$$\sigma_x = Y_i \left(\frac{\mathrm{d}u_i}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \right) - e_i \frac{\phi_0}{h_i}, \quad (x,y,z) \in B_i, \quad (i = 1,2). \tag{6}$$

We define the following stress resultants

$$N_i = \int_{A_i} \sigma_x \mathrm{d}A, \quad M_i = \int_{A_i} z\sigma_x \mathrm{d}A, \quad (i = 1,2). \tag{7}$$

By simple computations we obtain

$$N_i = A_i Y_i \left(\frac{\mathrm{d}u_i}{\mathrm{d}x} + (-1)^i c_i \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \right) - e_i b\phi_0, \quad (i = 1,2), \tag{8}$$

$$M_i = (-1)^{i+1} c_i A_i Y_i \frac{\mathrm{d}u_i}{\mathrm{d}x} - I_i Y_i \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + (-1)^i e_i c_i b\phi_0, \quad (i = 1,2). \tag{9}$$

Here, $I_i = \int\limits_{A_i} z^2 \mathrm{d}A \quad (i=1,2)$. There is no applied mechanical load in axial direction, this means that

$$N = N_1 + N_2 = 0. \tag{10}$$

The interlayer slip s is obtained as the difference of axial component of displacements defined on the common boundary of beam components B_1 and B_2, that is

$$s(x) = u_1(x) - u_2(x). \tag{11}$$

From Eqs. (10) and (11) we have

$$\frac{\mathrm{d}u_1}{\mathrm{d}x} = \frac{A_2Y_2}{\langle AY \rangle}\frac{\mathrm{d}s}{\mathrm{d}x} + \phi_0 \frac{b(e_1+e_2)}{\langle AY \rangle}, \tag{12}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}x} = -\frac{A_1Y_1}{\langle AY \rangle}\frac{\mathrm{d}s}{\mathrm{d}x} + \phi_0 \frac{b(e_1+e_2)}{\langle AY \rangle}. \tag{13}$$

Substituting of Eqs. (12), (13) into Eq. (8) gives

$$N_1 = \langle AY \rangle_{-1}\left[\frac{\mathrm{d}s}{\mathrm{d}x} - c\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right] - \frac{b}{c}(c_1e_1 - c_2e_2)\phi_0, \tag{14}$$

$$N_2 = \langle AY \rangle_{-1}\left[-\frac{\mathrm{d}s}{\mathrm{d}x} + c\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right] + \frac{b}{c}(c_1e_1 - c_2e_2)\phi_0. \tag{15}$$

Here,

$$\langle AY \rangle_{-1} = \frac{A_1Y_1A_2Y_2}{A_1Y_1 + A_2Y_2}. \tag{16}$$

From Eqs. (9), (12), (13) we can derive the expression of the total bending moment acting on the whole cross-section $A = A_1 \cup A_2$ as

$$M = M_1 + M_2 = c\langle AY \rangle_{-1}\frac{\mathrm{d}s}{\mathrm{d}x} - \{IY\}\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - (e_1c_1 - e_2c_2)b\phi_0, \quad \{IY\} = I_1Y_1 + I_2Y_2. \tag{17}$$

The whole cross-section is loaded by the shear force V which is

$$V = \frac{\mathrm{d}M}{\mathrm{d}x} = c\langle AY \rangle_{-1}\frac{\mathrm{d}^2s}{\mathrm{d}x^2} - \{IY\}\frac{\mathrm{d}^3w}{\mathrm{d}x^3}. \tag{18}$$

The interlayer shear force S acting in axial direction is a linear function of the slip, its expression is

$$S = ks, \tag{19}$$

where k is the slip modulus [3]. The force equilibrium equation for beam component B_1 can be formulated as

$$\frac{\mathrm{d}N_1}{\mathrm{d}x} - ks = \langle AY \rangle_{-1} \left[\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} - c \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} \right] - ks = 0. \tag{20}$$

Combination of Eq. (18) with Eq. (20) yields

$$\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} - \Omega^2 s + c \frac{V}{\langle IY \rangle} = 0, \tag{21}$$

where

$$\Omega^2 = k \frac{\{IY\}}{\langle IY \rangle \langle AY \rangle_{-1}}, \quad \langle IY \rangle = \{IY\} - c^2 \langle AY \rangle_{-1}. \tag{22}$$

In problem shown in Fig. 1. $V \equiv 0$ and $s(0) = 0$, thus we have for $s = s(x)$

$$s(x) = K \sinh \Omega x. \tag{23}$$

The constant K is obtained from the next boundary condition (Fig. 1.)

$$N_1(L) = 0. \tag{24}$$

2. DETERMINATION OF SLIP AND DEFLECTION

On the whole beam $M(x) = 0$. From this equation we get

$$\{IY\} \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = c \langle AY \rangle_{-1} \frac{\mathrm{d}s}{\mathrm{d}x} - (c_1 e_1 - c_2 e_2) b \phi_0. \tag{25}$$

Integration of Eq. (25) provides

$$\{IY\} \frac{\mathrm{d}w}{\mathrm{d}x} - \{IY\} \left(\frac{\mathrm{d}w}{\mathrm{d}x} \right)_0 = c \langle AY \rangle_{-1} \left(s(x) - s(0) \right) - \phi_0 b \left(c_1 e_1 - c_2 e_2 \right) x, \tag{26}$$

that is

$$\{IY\} \frac{\mathrm{d}w}{\mathrm{d}x} = c \langle AY \rangle_{-1} s(x) - \phi_0 b \left(c_1 e_1 - c_2 e_2 \right) x, \tag{27}$$

since $\left(\frac{\mathrm{d}w}{\mathrm{d}x} \right)_0 = 0$ (Fig. 1.). A repeated integration gives the result

$$\{IY\}\, w(x) = cK\langle AY\rangle_{-1}\frac{\cosh\Omega x - 1}{\Omega} - \phi_0 b\left(c_1e_1 - c_2e_2\right)\frac{x^2}{2}, \tag{28}$$

since $w(0)=0$. Lengthy but elementary computations yield

$$N_1(L) = \frac{\langle IY\rangle}{\{IY\}}\left[\langle AY\rangle_{-1}\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)_L - \frac{b}{c}\left(c_1e_1 - c_2e_2\right)\phi_0\right]. \tag{29}$$

Combination of Eq. (23) with Eq. (29) leads to the formula of K

$$K = \frac{b\left(c_1e_1 - c_2e_2\right)}{c\langle AY\rangle_{-1}\Omega\cosh\Omega L}\phi_0. \tag{30}$$

The final expression of the slip function can be written in the next form

$$s(x) = \phi_0\frac{b\left(c_1e_1 - c_2e_2\right)}{c\Omega\langle AY\rangle_{-1}}\frac{\sinh\Omega x}{\cosh\Omega L}. \tag{31}$$

Substituting K into equation (28) provides the deflection of two-layered piezoelectric beam in terms of applied voltage

$$w(x) = \phi_0\frac{b(c_1e_1 - c_2e_2)}{\{IY\}}\left[\frac{\cosh\Omega x - 1}{\Omega^2\cosh\Omega L} - \frac{x^2}{2}\right]. \tag{32}$$

For bimorph beam $h_1 = h_2$, $c_1 = c_2 = \frac{c}{2}$, $e_1 = -e_2 = e$ we have

$$s(x) = \phi_0\frac{eb}{\Omega\langle AY\rangle_{-1}}\frac{\sinh\Omega x}{\cosh\Omega L}, \tag{33}$$

$$w(x) = \phi_0\frac{ecb}{\{IY\}}\left[\frac{\cosh\Omega x - 1}{\Omega^2\cosh\Omega L} - \frac{x^2}{2}\right]. \tag{34}$$

3. NUMERICAL EXAMPLE

The following data have been used in numerical computations: $h_1 = h_2 = 0.002$ $[\mathrm{m}]$, $b = 0.005$ $[\mathrm{m}]$, $L = 0.5$ $[\mathrm{m}]$, $Y_1 = Y_2 = 50$ $[\mathrm{GPa}]$, $e_1 = e_2 = -17.6$ $[\mathrm{N/mV}]$, $\phi_0 = 200$ $[\mathrm{V}]$. The slip functions and deflection functions for $k_1 = 10^2$ $[\mathrm{Pa}]$, $k_2 = 10^4$ $[\mathrm{Pa}]$, $k_3 = 10^6$ $[\mathrm{Pa}]$, $k_4 = 10^8$ $[\mathrm{Pa}]$, $k_5 = 10^{10}$ $[\mathrm{Pa}]$ are determined. The graphs of slip and deflection functions for k_1, k_2, k_3, k_4 and k_5 are shown in Fig. 2. and Fig. 3., respectively.

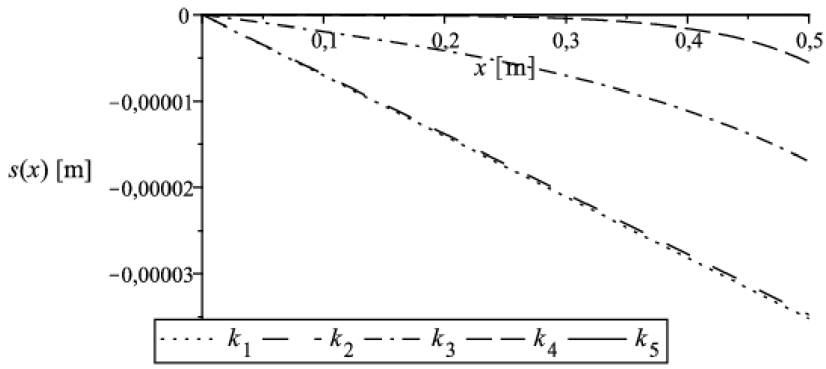

Figure 2. Plots of slip functions.

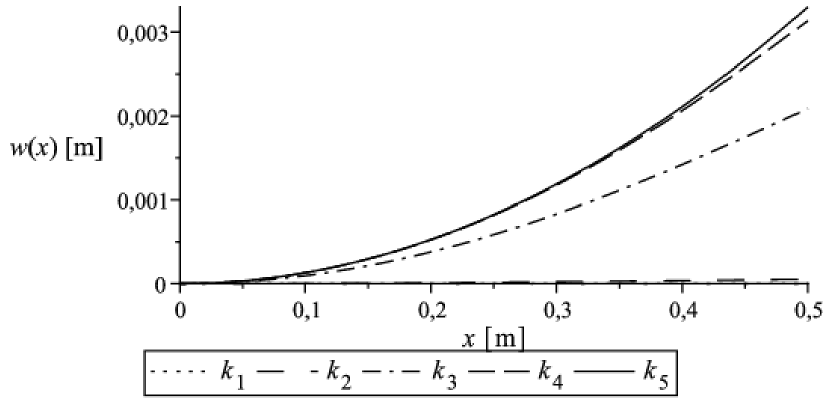

Figure 3. Plots of deflection functions.

4. CONCLUSIONS

This paper presents an analytical solution for two-layered piezoelectric cantilever beam with imperfect shear connection. Simple formulas are derived to obtain the deflection and slip functions. An example illustrates the application of derived formulas and the effect of slip modulus to the deformation and slip.

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