

SOLUTION OF ELASTIC CONTACT PROBLEMS BY THE FINITE ELEMENT DISPLACEMENT METHOD

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Contact problem of elastic continua subject to arbitrary load and of arbitrary surface is a rather intricate one, contact domains not being known a priori. Here, the continuum is replaced by a bulk of elements of finite degrees of freedom, the obtained elastic system serving as basis for the solution of the problem based on the principle of potential energy minimum. Because of the unilateral relations between the bodies, the mathematical programming can be discussed as a quadratic programming problem. Use of the Khun-Tucker conditions yields a solution for the dual of this primal problem, much easier to establish and solve than the original one. Friction and adherence between the bodies are considered as negligible, and displacements, deformations to be small.

Introduction

With the advent and generalization of computers, elasticity and mathematical methods easy to computerize have been developed, such as the method of finite elements and various methods of mathematical programming.

In spite of the approximation involved in the combined application of both methods, its practical significance must not be underestimated, since the complexity of mechanical or architectural structures, of various types of load practically prevents any exact solution.

A practical satisfactory solution of the contact problem is possible by discretizing the finite elements according to size and kind, and applying the so-called condition of the contact/separation discussed in item 1.2.2.2.

According to the Author's knowledge, the first publications on the mathematical programming of contact problems date back to 1967. An interesting, universal, efficient method has been suggested by FRIEDMANN, V. M. and TSCHERNINA, V. S. [1], [2], i. e. cyclic iteration based on the gradient method, to approximate the first-kind integral equation-inequality describing the contact-separation phenomenon; and that with a universal formulation, suitable for examining a wide range of structures. As an example, solution of contact problems of coaxial cylinders and rings, as well as of a circular plate under symmetric load and a Winkler type foundation is described in [2].

Theoretical bases and a concrete solution method for the calculation of

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elastically an bedded structures — involving the method of quadratic programming — has been considered in the significant, pioneering work by DUPUIS, G. and PROBST, A. [3]; potential energy of the system is minimalized under the constraint expressing the conditions of the contact/separation at a finite number of points. The elastic strain energy of the foundation and the beam contacting is approximated by the method of differences.

For the sake of completeness, let us mention references [4] to [7] concerning the theoretical and practical problems of unilateral relations occurring at structure supports — although these do not strictly belong to the scope of this study.

Thesis [8] suggests developing ideas in [2] permitting the consideration, in addition to rigid-body displacements, also of angular rotation between solids, what is more, a subsequent paper [9] accepts — in addition to contact forces, — internal forces developing at structural joints (e. g. of shells, plates) as unknowns.

[10] and [11] are concerned with the contact problem of advancing bodies by means of the method of quadratic programming assuming that within the proposed zone of the contact, the system of (other than contact) forces acting on the bodies causes no displacement, and that the relative rigid body translation between the bodies can only be positive, i. e. an approach. Solutions in [1] and [9] are exempt from these restrictions.

The problem of contact between the beam or plate and a Winkler-type foundation is solved in [12] and [13], so as to minimize the potential energy by the constants involved in the series expansion of the deflection functions, keeping in mind the conditions of the contact separation; the solution is obtained by quadratic programming.

A similar concept is encountered in [14] for the solution of the contact between a symmetrically loaded circular plate and the elastic semi-space.

Several works by CONWAY, H. D. and ENGEL, P. A. have been concerned with contact problems in cold rolling. Among them, [15] analyses the contact between the stiff cylinder and the elastic layer; the contact separation condition is checked at a finite number of points; step-wise load increments are determined by involving ever more points in the contact. The algebraic equation-inequality system expressing the contact separation is essentially the same as that for the approximate solution of the Fredholm integral equation inequality encountered on setting up this problem. In this relation it is affine to that by FRIEDMANN, V. M., TSCHERNINA, V. S., while solution steps of the algebraic equation inequality system are quite different. Another paper by them [16] also reckons with the effect of friction.

Along with the popularization of the finite element method, several papers were published on the calculation of various elastic supports ([17], [18], [19]), *considering the connection between the bodies as bilateral.*

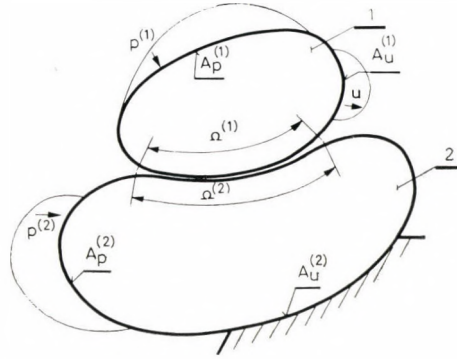


Fig. 1. The problem is to find the real contact domain Ω_p between bodies 1 and 2, displacement vector field and stress state of the bodies

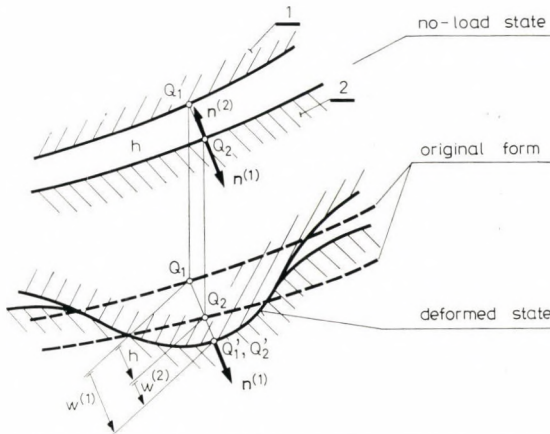


Fig. 2. Geometry conditions of contact separation:

$$\begin{aligned} &\text{for } w^{(2)} - w^{(1)} + h = 0 \text{ contact,} \\ &\text{for } w^{(2)} - w^{(2)} + h > 0 \text{ separation (gap)} \end{aligned}$$

A quite different approach is encountered in [27], also taking the Coulomb friction between the bodies into account; separation and contact domains can be determined by gradually increasing the load and solving the problem step-wise. Convergency of the method is, however, questionable.

[20] presents a contact problem between a rigid sphere and an ideally plastic semi-space.

Slide ways of tool machines are analysed by the finite element method in [21], making use of the empirical relationship between deflection and pressure, depending on the design, to determine the contact pressure. The method suggested in this study eliminates the empirical relationship.

After this short survey of literature — with no claim to completeness — let us consider the contact problem itself.

For the sake of simplicity, let us assume that the elastic system consists of two bodies, with no restriction for the generality of the problem, at the same time facilitating to expose, set out and simply treat peculiar problems of contact.

Let the bodies in Fig. 1 be denoted by $t = 1, 2$; assignment of any magnitude will be indicated by the superscript in parentheses. Body surfaces will be separated into three domains:

$A_p^{(t)}$ — body surface part bearing a given surface load;

$A_u^{(t)}$ — body surface part with given displacements (kinematic boundary conditions);

$\Omega^{(t)}$ — proposed zone of contact;

$A^{(t)} = A_p^{(t)} + A_u^{(t)} + \Omega^{(t)}$ — body surface.

Coupling each two points of body surfaces into domain $\Omega^{(t)}$ their displacements or a component in some direction e. g. normally will be represented in course of the deformation¹ (Fig. 2.).

Correlation between points in domains $\Omega^{(1)}$ and $\Omega^{(2)}$ being settled, in the following the superscript will be omitted, writing Ω alone.

Unilateral relations between bodies implies that for a difference

$$n^{(1)r}(u_r^{(2)} - u_r^{(1)} + h_r) = w^{(1)} - w^{(2)} + h \equiv y > 0, \quad (1)$$

between the projections of the displacement vectors² $u_r^{(t)}$ of points in domain Ω in a given direction (e. g. the outer normal of body 1 at point Q_1) separation occurs, that is, the contact force is zero, $p = 0$, and for

$$n^{(1)r}(u_r^{(2)} - u_r^{(1)} + h_r) = w^{(2)} - w^{(1)} + h \equiv y = 0, \quad (2)$$

the body point couples contact each other, hence $p \geq 0$. Let the separation gap domain be denoted by $x \in \Omega_0$, the contact domain by $x \in \Omega_p$, and be $\Omega = \Omega_0 + \Omega_p$, where x is a tridimensional co-ordinate in spatial problems, a two-dimensional one for planar problems, and a linear one for single-variable problems; $n^{(1)r}$ — is the outer normal of body 1; $u_r^{(t)}$ the displacement vector of body t ; and h_r the vector of the initial gap. By definition, $w^{(t)}$ is the projection of the displacement of a point in domain Ω in the indicated direction, furthermore, domains Ω_0 and Ω_p are unknown.

Because of the unilateral connections, for points in domain Ω :

$$yp = 0, \quad x \in \Omega. \quad (3)$$

¹ The same hypothesis is encountered in classic solutions of rigid-punch semi-space semi-plane contact problems assuming small displacements (see e. g. in [22]).

² Vectors and tensors with covariant or contravariant components are denoted by subscripts and superscripts, as is usual. Latin letters may assume values of 1,2,3 [22].

1. Solution of contact problems according to the principle of potential energy minimum

1.1 Setting the problem

To solve the contact problem, the displacement vector field of bodies belonging to the elastic system will be approximated by the finite element method, also taking the interaction of bodies due to unilateral connections (contact effect) into consideration, in determining the unknown parameters. This approximation is especially advantageous for bodies complex in form, difficult to clamp, and subject to combined loads; inconvenient, practically impossible to solve.

Let us assume a priori a bilateral connection between the bodies in the indicated direction $n_r^{(1)}$, that is:

$$w^{(2)} - w^{(1)} + h = 0, \quad x \in \Omega. \quad (4)$$

The displacement field $u_r^{(t)}$ to be approximated is required to meet kinematic boundary conditions on surface $A_u^{(t)}$. Then the potential energy of body t (for zero initial deformation and stress):

$$\pi^{(t)} = \frac{1}{2} \int_{(V^{(t)})} C^{klrs} a_{kl} a_{rs} dV - \int_{(V^{(t)})} q^s u_s dV - \int_{(A_p^{(t)})} p_o^s u_s dA, \quad t = 1, 2 \quad (5)$$

where $V^{(t)}$ — volume of body t ; $A_p^{(t)}$ — its surface under prescribed surface traction p^s ; C^{klrs} — matrix of material constants; a_{rs} — strain tensor; q^s — volumetric force intensity vector, $\sigma^{rs} = C^{rskl} a_{kl}$ — stress tensor according to Hooke's law.

Introducing functional

$$L_1 = L_1(u_r^{(1)}, u_r^{(2)}, p) = \pi^{(1)} + \pi^{(2)} + \int_{(\Omega)} p(w^{(1)} - w^{(2)} - h) dA \quad (6)$$

in view of $a_{rs} = [(u_r \nabla_s) + (\nabla_r u_s)]/2$, for small displacements, involving the Hamilton differential operator ∇_s ; and $C^{klrs} = C^{rskl}$, displacement field i. e. stress $n_r^{(t)} \sigma^{(t)rs}$ in points of domain Ω can be separated into components in a given direction $n^{(1)r}$ and normally to it, then, from Hooke's law and the Gauss—Ostrogradsky integral transformation theorem, the following conclusions of the Euler equations belonging to the stationary value of the functional ($\delta L_1 = 0$) are obvious, assuming the relation to be bilateral:

1. the Lamé equilibrium equation in terms of the displacement vector:

$$\nabla_k \sigma^{(t)kl}(u_s^{(t)}) + q^{(t)l} = 0, \quad x \in V^{(t)};$$

2. fulfilment of the dynamic boundary condition:

$$n_r^{(t)} \sigma^{(t)rs} = p_o^s, x \in A_p^{(t)};$$

3. fulfilment of the supplementary kinematic boundary condition (4) related to both bodies, arising from their bilateral relation;

4. relationship for the contact force as an internal force:

$$p = - n_r^{(1)} \sigma^{(1)rs} n_s^{(1)}, \quad p = n_r^{(1)} \sigma^{(2)rs} n_s^{(2)};$$

5. zero tangential stresses from neglecting the friction and adherence

$$\tau^{(1)} = | n_r^{(1)} \sigma^{(1)rs} n^{(1)p} \varepsilon_{spv} | = 0,$$

$$\tau^{(2)} = | n_r^{(2)} \sigma^{(2)rs} n^{(1)p} \varepsilon_{spv} | = 0,$$

where ε_{rst} is the Levi-Civita tensor [22].

Change of functional (6):

$$\Delta L_1 = \delta L_1 + \delta^2 L_1 = \delta^2 \pi + \int_{(\Omega)} \delta_p (\delta w^{(1)} - \delta w^{(2)}) dA,$$

since $\delta L_1 = 0$, while according to $\delta^2 \pi > 0$, $\pi = \pi^{(1)} + \pi^{(2)}$ is a potential energy with a minimum.

Taking note that, because of (4), in case of an exact solution

$$L_1 = \pi^{(1)} + \pi^{(2)}. \quad (7)$$

This relationship is also valid in case of a unilateral relation between the bodies, hence, where equality (4) is replaced by inequality-equality (1), (2) and expression (3). Since the unilateral relation leads to $p \geq 0$, $y \geq 0$, $y \cdot p = 0$, the problem has to be developed so as to permit application of mathematical programming methods.

1.2. Treatment of the contact problem by quadratic programming

From the mentioned aspect, the problem has to be discretized; partly, the displacement vector field of the bodies will be approximated by a kinematically admissible displacement field, and partly, the integral value from the constraint over the domain Ω involved in the functional L_1 (see in (6)) will be so approximated that it appears as a finite sum.

1.2.1 Assumption of the displacement field to be approximated

Kinematically possible displacement fields are required to meet the kinematical boundary conditions on the surface $A_u^{(i)}$ and to deliver the strain tensor. The displacement vector field could also be approximated in series form according to the Ritz method, but because of the important change of the stress tensor field adjacent to the contact domains, as well as the complicated form and load of the bodies constituting the elastic system, it seems more expedient to apply the so-called compatible displacement model of the finite element method³ or of any other model expressing the potential energy of the system obtained by transforming the functional, to be minimized in terms of the generalized nodal displacement vector, such as models of hybrid displacement, by P. TONG [24] and by T. H. H. PIAN [25] of hybrid stress and mixed fields.

In the compatible displacement model of the finite elements method, the kinematically admissible displacement field is approximated by the entity of functions assumed to be element-wise [26]. Let the displacement field be⁴ inside element e obtained by separating the body denoted by $\mathbf{u}^e(x)$, and its (generalized) nodal displacements by \mathbf{u}^e . In terms of the assumed approximation matrix $\mathbf{A}^e(x)$, the displacement field inside the element is:

$$\mathbf{u}^e(x) = \mathbf{A}^e(x) \mathbf{u}^e. \quad (8)$$

Denoting vectors formed of members of the strain and stress tensor by $\boldsymbol{\epsilon}^e(x)$ and $\boldsymbol{\sigma}^e(x)$ resp., and in knowledge of the displacement field:

$$\boldsymbol{\epsilon}^e(x) = \mathbf{D} \mathbf{u}^e(x) = \mathbf{B}^e(x) \mathbf{u}^e \quad (9)$$

where \mathbf{D} — differential operator matrix, while making use of Hooke's law and assuming initial deformations and stresses to be zero,

$$\boldsymbol{\sigma}^e(x) = \mathbf{C}^e \boldsymbol{\epsilon}^e(x) = \mathbf{C}^e \mathbf{B}^e(x) \mathbf{u}^e \quad (10)$$

where \mathbf{C}^e — is the matrix of material constants.

Applying the compatible displacement model, the assumed approximation matrix $\mathbf{A}^e(x)$ must yield strain $\boldsymbol{\epsilon}^e(x) \equiv 0$ if the element performs a rigid body motion, further, the transition from one element to the other has to respect the displacement continuity [26].

Introducing the above magnitudes in (5) yields the potential energy for element e :

$$\pi_e = \frac{1}{2} \mathbf{u}^{e,T} \mathbf{K}^e \mathbf{u}^e - \mathbf{q}_*^{e,T} \mathbf{u}^e, \quad (11)$$

³ Bases of the finite element method are assumed to be known.

⁴ Multidimensional vectors and matrices are marked by bold letters, and their transposition by the right superscripts T .

where

$$\mathbf{K}^e = \int_{(V^e)} \mathbf{B}^{e,T}(\mathbf{x}) \mathbf{C}^e \mathbf{B}^e(\mathbf{x}) dV - \text{the stiffness matrix of the element;}$$

$$\text{and} \quad \begin{cases} \tilde{\mathbf{q}}^e = \int_{(V^e)} \mathbf{A}^{e,T}(\mathbf{x}) \mathbf{q}^e(\mathbf{x}) dV \\ \tilde{\mathbf{p}}^e = \int_{(A_p^e)} \mathbf{A}^{e,T}(\mathbf{x}) \mathbf{p}^e(\mathbf{x}) dA \end{cases} \left\{ \begin{array}{l} \text{generalized nodal force val-} \\ \text{ues from volumetric and} \\ \text{surface loads } \mathbf{q}^e(\mathbf{x}) \text{ and} \\ \mathbf{p}^e(\mathbf{x}), \text{ resp., and:} \end{array} \right.$$

$$\mathbf{q}_*^e = \tilde{\mathbf{q}}^e + \tilde{\mathbf{p}}^e.$$

Making use of the identity of the generalized displacement vector at identically marked nodes of adjacent elements, permitting the production of potential energy for the tested body:

$$\pi^{(t)} = \frac{1}{2} \tilde{\mathbf{u}}^{(t),T} \tilde{\mathbf{K}}^{(t)} \tilde{\mathbf{u}}^{(t)} - \tilde{\mathbf{u}}^{(t),T} \tilde{\mathbf{q}}_*^{(t)}, \quad t = 1, 2 \quad (12a)$$

where:

- $\tilde{\mathbf{u}}^{(t)}$ — generalized nodal displacement vector for body t ;
- $\tilde{\mathbf{K}}^{(t)}$ — stiffness matrix ($\tilde{\mathbf{K}}^{(t)} = \tilde{\mathbf{K}}^{(t),T}$);
- $\tilde{\mathbf{q}}_*^{(t)}$ — generalized nodal load vector from external loads.

The value of the displacement vector field at nodes on the body surface $x \in A_u^{(t)}$ being known, $\tilde{\mathbf{u}}^{(t)}$ can be separated into two parts, i. e. vectors with unknown and with known components: $\tilde{\mathbf{u}}^{(t),T} = [\mathbf{u}^{(t),T} \mid \hat{\mathbf{u}}^{(t),T}]$; and so can be $\tilde{\mathbf{q}}_*^{(t)}$: $\tilde{\mathbf{q}}_*^{(t),T} = [\bar{\mathbf{q}}^{(t),T} \mid \hat{\mathbf{q}}^{(t),T}]$. Then:

$$\pi^{(t)} = \frac{1}{2} [\mathbf{u}^{(t),T} \mid \hat{\mathbf{u}}^{(t),T}] \begin{bmatrix} \mathbf{K}^{(t)} & \tilde{\mathbf{K}}_{12}^{(t)} \\ \tilde{\mathbf{K}}_{21}^{(t)} & \tilde{\mathbf{K}}_{22}^{(t)} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(t)} \\ \hat{\mathbf{u}}^{(t)} \end{bmatrix} - [\mathbf{u}^{(t),T} \mid \hat{\mathbf{u}}^{(t),T}] \begin{bmatrix} \bar{\mathbf{q}}^{(t)} \\ \hat{\mathbf{q}}^{(t)} \end{bmatrix}$$

and after performing the operations, and taking into consideration that in $\tilde{\mathbf{u}}^{(t)}$ the only unknown is $\mathbf{u}^{(t)}$, for the functional to be constructed, $\pi^{(t)}$ may be simply understood as:

$$\pi_*^{(t)} = \frac{1}{2} \mathbf{u}^{(t),T} \mathbf{K}^{(t)} \mathbf{u}^{(t)} - \mathbf{u}^{(t),T} \mathbf{q}^{(t)} = \pi^{(t)} - \text{const} \quad (12b)$$

where

$$\mathbf{q}^{(t)} = \bar{\mathbf{q}}^{(t)} - \tilde{\mathbf{K}}_{12}^{(t)} \hat{\mathbf{u}}^{(t)} - \text{generalized nodal load vector from the known external load system and the displacement vector } \hat{\mathbf{u}}^{(t)}.$$

1.2.2 Discretization of the constraint

Let us approximate the integral value in functional L_1 :

1.2.2.1 Checking contact/separation in discrete points:

$$\begin{aligned} \int_{(\Omega)} p(w^{(1)} - w^{(2)} - h) dA &= \int_{(\Omega)} p(x)(w^{(1)}(x) - w^{(2)}(x) - h(x)) dA \cong \\ &\cong \sum_{i=1}^k [p(x_i) \Delta A_i] (w^{(1)}(x_i) - w^{(2)}(x_i) - h(x_i)) = \\ &= \sum_{i=1}^k P_i (w_i^{(1)} - w_i^{(2)} - h_i) = - \sum_{i=1}^k P_i y_i, \end{aligned} \quad (13)$$

introducing notations

$$\begin{aligned} P_i &= p(x_i) \Delta A_i \geq 0, \quad w^{(t)}(x_i) = w_i^{(t)}, \quad t = 1, 2, \\ y_i &= w_i^{(2)} - w_i^{(1)} + h_i \geq 0; \end{aligned}$$

non-negativity of P_i and y_i follows from the unilateral relations (see in (1) through (3)).

Integral approximation can be mechanically interpreted as follows: domain Ω is separated into k finite small parts ΔA_i , the contact force distributed over them is replaced by its "resultant" P_i realized at an inner point (e. g. center) of the given domain part ΔA_i , and other terms of the integrand are calculated at these inner points.

Thereby the contact separation condition is checked at k points. Increase of k and a more exact calculation of integral (13) means the increase in accuracy by solving the contact problem.

Introducing the resultant vector of the contact surface stress

$$\bar{\mathbf{p}}^T = [P_1, P_2, \dots, P_k] \geq \mathbf{0} \quad (14)$$

and vectors of size $(1 \times k)$

$$\mathbf{w}^{(t),T} = [w_1^{(t)}, w_2^{(t)}, \dots, w_k^{(t)}], \quad t = 1, 2 \quad (15)$$

$$\bar{\mathbf{h}}^T = [h_1, h_2, \dots, h_k], \quad (16)$$

$$\bar{\mathbf{y}}^T = [y_1, y_2, \dots, y_k] \geq \mathbf{0} \quad (17)$$

construction of a corresponding permuting matrix may yield:

$$\mathbf{w}^{(t)} = \mathbf{G}^{(t)} \mathbf{u}^{(t)}, \quad t = 1, 2 \quad (18)$$

hence, in view of (3), the constraint (13) may be written more concisely:

$$\int_{(\Omega)} p(w^{(1)} - w^{(2)} - h) dA = \bar{\mathbf{p}}^T [\mathbf{G}^{(1)} \mathbf{u}^{(1)} - \mathbf{G}^{(2)} \mathbf{u}^{(2)} - \bar{\mathbf{h}}], \quad (19a)$$

$$\bar{\mathbf{p}}^T \bar{\mathbf{y}} = \mathbf{0}; \quad \bar{\mathbf{p}} \geq \mathbf{0}; \quad \bar{\mathbf{y}} \geq \mathbf{0}. \quad (19b, c, d)$$

1.2.2.2 Checking contact/separation in small finite sections

The previous approximation is valid for a displacement field inside the element, considered as varying, according to a linear law. Now, the stress state of the element is constant (in a spatial case it is a tetrahedron of 12 degrees of freedom, in plane problems triangles of 6 degrees of freedom), i. e., the generalized nodal force due to contact force p considered as constant becomes a concentrated force (resultant) of direction $w^{(t)}$.

In cases where the selected element type approximates the displacement field inside the element by polynomials higher than of first degree, the generalized nodal force due to the contact force developed on the element surface in domain Ω can only be calculated from the relationship

$$\tilde{\mathbf{p}}^e = \int_{(A_\Omega^e)} \mathbf{A}^{eT}(x) \mathbf{p}^e(x) dA.$$

The initially unknown contact force $p^e(x)$ cannot be approximated by a polynomial of higher degree than the selected element type is able to yield. If this fact is respected, our calculation will comply with the relationship for "the contact force as internal force" belonging to the stationary value of the functional L_1 .

Three points of view support the use of element types delivering a non-constant stress field inside the element, viz.:

1. according to computational experience, applying the element type of more degrees of freedom for approximating the stress state at the same accuracy leads to far less unknowns in the final set of equations than for the simpler elements;

2. stress state approximation is improved, a major requirement especially near domain Ω ;

3. not too small elements have to be handled near the domain Ω , either.

Assuming the approximation matrices $\tilde{\mathbf{A}}^{(t)}(x)$ (row vectors) to be produced by using matrices $\mathbf{A}^e(x)$ for approximating displacements $w^{(t)}(x)$ ($t = 1, 2$) along the indicated direction in domain Ω . If the generalized displacement vector $\tilde{\mathbf{u}}^{(t)}$ is known:

$$w^{(t)}(x) = \tilde{\mathbf{A}}^{(t)}(x) \tilde{\mathbf{u}}^{(t)} = \mathbf{A}^{(t)}(x) \mathbf{u}^{(t)} + \hat{\mathbf{A}}^{(t)}(x) \hat{\mathbf{u}}^{(t)}, \quad x \in \Omega. \quad (20)$$

The initial gap can be written by means of vector h composed of approximation matrix $\mathbf{L}(x)$ (row vector) and its \mathbf{h} values at discrete points:

$$h(x) = \mathbf{L}(x) \mathbf{h}. \quad (21)$$

Accordingly, the contact force is approximated in the form:

$$p(x) = \mathbf{P}^T(x) \mathbf{p}, \quad x \in \Omega \quad (22)$$

where

$\mathbf{P}^T(x)$ — approximation matrix (row vector) depending on the element type;

\mathbf{p} — a vector produced from the contact surface stresses developing at a finite number of points but not absolutely element nodes $[\text{N/cm}^2]$.

Now:

$$\int_{(\Omega)} p[w^{(1)} - w^{(2)} - h] dA \cong \mathbf{p}^T [\mathbf{A}^{(1)} \mathbf{u}^{(1)} - \mathbf{A}^{(2)} \mathbf{u}^{(2)} - \mathbf{b}], \quad (23a)$$

where

$$\mathbf{A}^{(t)} = \int_{(\Omega)} \mathbf{P}(x) \mathbf{A}^{(t)}(x) dA, \quad t = 1, 2 \quad (23b)$$

$$\begin{aligned} \mathbf{b} &= \int_{(\Omega)} \mathbf{P}(x) \mathbf{L}(x) dA \mathbf{h} + \\ &+ \int_{(\Omega)} \mathbf{P}(x) \hat{\mathbf{A}}^{(2)}(x) dA \hat{\mathbf{u}}^{(2)} - \\ &- \int_{(\Omega)} \mathbf{P}(x) \hat{\mathbf{A}}^{(1)}(x) dA \hat{\mathbf{u}}^{(1)}. \end{aligned} \quad (23c)$$

Introducing magnitudes

$$\mathbf{A} \underset{k \times (m+n)}{=} = \left[\begin{array}{c} \mathbf{A}^{(1)} \\ \vdots \\ \mathbf{A}^{(2)} \end{array} \right] \underset{(k \times n)}{=} \quad (24a)$$

$$\mathbf{u} \underset{(m+n) \times 1}{=} = \left[\begin{array}{c} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(2)} \end{array} \right] \underset{(n \times 1)}{=} \quad (24b)$$

$$\mathbf{y}^{(0)} \underset{(k \times 1)}{=} \equiv \mathbf{A} \mathbf{u} - \mathbf{b} \leq \mathbf{0} \quad (24c)$$

permits the production of the constraint in the form:

$$\int_{(\Omega)} p[w^{(1)} - w^{(2)} - h] dA \cong \mathbf{p}^T [\mathbf{A} \mathbf{u} - \mathbf{b}] \quad (25a)$$

$$\mathbf{p} \geq \mathbf{0}, \mathbf{y}^{(0)} \geq \mathbf{0}, \mathbf{p}^T \mathbf{y}^{(0)} = \mathbf{0}. \quad (25b, c, d)$$

1.2.3 Formulation of the quadratic programming problem

1.2.3.1 Primal problem

Constraints obtained by transformations described in items 1.2.2.1 and 1.2.2.2 were seen to be formally identical but physically, qualitatively

different. Further computations will be based on constraint expressed by relationships (25a–d) in item 1.2.2.2 but statements are also valid for constraint in 1.2.2.1.

Solution of the examined contact problem was proved in item 1.1 to be provided for by the stationary position of functional L_1 , provided bodies are bilaterally related. Because of the constraint (25a–d) resulting from unilateral relations, and of the kinematically possible displacement field approximated by the finite element method, the functional is modified, that is, applying (12b), (24a, b) and (25a), the initial L_1 is replaced by L_2 :

$$L_1 = L_1(u_r^{(1)}, u_r^{(2)}, p) \rightarrow L_2 = L_2(\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{p}),$$

that is

$$L_2 = \frac{1}{2} [\mathbf{u}^{(1),T} \mid \mathbf{u}^{(2),T}] \begin{bmatrix} \mathbf{K}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \end{bmatrix} - [\mathbf{u}^{(1),T} \mid \mathbf{u}^{(2),T}] \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \end{bmatrix} + \mathbf{p}^T ([\mathbf{A}^{(1)} \mid -\mathbf{A}^{(2)}] \begin{bmatrix} \mathbf{u}^{(1)} \\ \mathbf{u}^{(2)} \end{bmatrix} - \mathbf{b}) \quad (26)$$

or, more concisely:

$$L_2 = L_2(\mathbf{u}, \mathbf{p}) = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{q} + \mathbf{p}^T (\mathbf{A} \mathbf{u} - \mathbf{b}) = \pi(\mathbf{u}) + \mathbf{p}^T \mathbf{y}^{(0)}, \quad (27)$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{(2)} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \end{bmatrix}, \quad (28a, b)$$

$$\mathbf{p} \geq \mathbf{0}, \quad \mathbf{y}^{(0)} \geq \mathbf{0}, \quad \mathbf{p}^T \mathbf{y}^{(0)} = 0. \quad (29a, b, c)$$

Assuming a rigid-body-like relative displacement between solids to be possible, matrix \mathbf{K} is positive semidefinite ($\mathbf{x}^T \mathbf{K} \mathbf{x} \geq 0$, for $\mathbf{x} \neq \mathbf{0}$), that is, potential energy $\pi(\mathbf{u})$ of the system is a strictly quasi-convex quadratic function of the generalized displacement vector \mathbf{u} .

Thereby, the contact problem could be reduced to the following quadratic programming problem:

$$\min \left\{ \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{q} \mid \mathbf{A} \mathbf{u} - \mathbf{b} \leq \mathbf{0} \right\} \quad (30)$$

(27) can be considered as a Lagrange function of programming problem (30).

1.2.3.2 Dual problem

Theory of mathematical programming has demonstrated that the primal problem can be assigned a dual problem, and that the existence of solution

for one involves that for the other. Often — in actual computations — solution of the dual problem, followed by the determination of unknowns in the original problem, seems more practical than finding a direct solution for the primal problem.

In this case, the dual problem belonging to (30) can be produced by making use of the Khun—Tucker theorems of non-linear programming (see in the Appendix).

The theorem leads to relationships

$$\frac{\partial L_2}{\partial \mathbf{u}^{(1)}} = \mathbf{K}^{(1)} \mathbf{u}^{(1)} - \mathbf{q}^{(1)} + \mathbf{A}^{(1),T} \mathbf{p} = \mathbf{0}, \quad (31a)$$

(according to (A.8) where $\mathbf{x} = \mathbf{u}^{(t)}$, $t = 1, 2$)

$$\frac{\partial L_2}{\partial \mathbf{u}^{(2)}} = \mathbf{K}^{(2)} \mathbf{u}^{(2)} - \mathbf{q}^{(2)} - \mathbf{A}^{(2),T} \mathbf{p} = \mathbf{0}, \quad (31b)$$

further:

$$\frac{\partial L_2}{\partial \mathbf{p}} = \mathbf{A}^{(1)} \mathbf{u}^{(1)} - \mathbf{A}^{(2)} \mathbf{u}^{(2)} - \mathbf{b} = -\mathbf{y}^{(0)} \leq \mathbf{0} \quad (32a)$$

$$\mathbf{p}^T \frac{\partial L_2}{\partial \mathbf{p}} = \mathbf{p}^T \mathbf{y}^{(0)} = 0 \quad (32b)$$

(according to (A.7) where $\mathbf{u} = \mathbf{p}$).

Assuming body 2 not performing rigid-body motion, $\det \mathbf{K}^{(2)} \neq 0$, thus, from (31b):

$$\mathbf{u}^{(2)} = [\mathbf{K}^{(2)}]^{-1} \mathbf{q}^{(2)} + [\mathbf{K}^{(2)}]^{-1} \mathbf{A}^{(2),T} \mathbf{p}. \quad (33)$$

Body 1 can perform rigid-body motion, hence, its stiffness matrix is a singular one, but a non-singular quadratic matrix of the size of the original matrix lessened by the degrees of freedom of the rigid-body motion can always be designated to it. Assuming matrix $\mathbf{K}_{22}^{(1)}$ in the lower right corner of matrix $\mathbf{K}^{(1)}$ not to be singular. (This is always possible by duly rearranging rows and columns.) Partitioning Eq. (31a) yields:

$$\begin{bmatrix} \mathbf{K}_{11}^{(1)} & \mathbf{K}_{12}^{(1)} \\ \mathbf{K}_{21}^{(1)} & \mathbf{K}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{(1)} \\ \mathbf{u}_{11}^{(1)} \end{bmatrix} - \begin{bmatrix} \mathbf{q}_1^{(1)} \\ \mathbf{q}_{11}^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_1^{(1),T} \\ \mathbf{A}_{11}^{(1),T} \end{bmatrix} \mathbf{p} = \mathbf{0} \quad (34)$$

thus, partly

$$\mathbf{u}_{11}^{(1)} = [\mathbf{K}_{22}^{(1)}]^{-1} \mathbf{q}_{11}^{(1)} - [\mathbf{K}_{22}^{(1)}]^{-1} \mathbf{A}_{11}^{(1),T} \mathbf{p} - [\mathbf{K}_{22}^{(1)}]^{-1} \mathbf{K}_{21}^{(1)} \mathbf{u}_1^{(1)} \quad (35)$$

$$\mathbf{F}^{(2)} = \underset{(k \times n)}{\mathbf{A}^{(2)}} \underset{(n \times n)}{[\mathbf{K}^{(2)}]^{-1}}, \quad \mathbf{f}^{(2)} = \underset{(k \times 1)}{\mathbf{F}^{(2)}} \underset{(n \times 1)}{\mathbf{q}^{(2)}}, \quad (40c, d)$$

$$\underset{(k \times 1)}{\mathbf{t}} = \mathbf{f}^{(1)} - \mathbf{f}^{(2)} - \mathbf{b}, \quad (40e)$$

$$\underset{(k \times k)}{\mathbf{H}^{(1)}} = \underset{[k \times (m-l)]}{\mathbf{A}_{11}^{(1)}} \underset{[(m-l) \times (m-l)]}{[\mathbf{K}_{22}^{(1)}]^{-1}} \underset{[(m-l) \times k]}{\mathbf{A}_{11}^{(1),T}}, \quad (41a)$$

$$\underset{(k \times k)}{\mathbf{H}^{(2)}} = \underset{(k \times n)}{\mathbf{A}^{(2)}} \underset{(n \times n)}{[\mathbf{K}^{(2)}]^{-1}} \underset{(n \times k)}{\mathbf{A}^{(2),T}}, \quad (41b)$$

$$\mathbf{H} = \mathbf{H}^{(1)} + \mathbf{H}^{(2)}, \quad (41c)$$

into consideration, geometrical equation inequality (39) expressing contact separation can be replaced by

$$\mathbf{H} \mathbf{p} - \mathbf{G} \mathbf{u}_1^{(1)} - \mathbf{t} = \mathbf{y}^{(0)} \geq \mathbf{0} \quad (42a)$$

and from (29c)

$$\mathbf{p}^T \mathbf{y}^{(0)} = 0 \quad (42b)$$

\mathbf{H} being the resultant "influence coefficient matrix", \mathbf{t} the displacement from known loads and the initial gap, and \mathbf{G} a matrix also involving the structure geometry.

Equations of equilibrium (37) and of geometry (42a) yield a hypermatrix equation

$$\underbrace{\begin{bmatrix} \mathbf{D} & \mathbf{G}^T \\ -\mathbf{G} & \mathbf{H} \end{bmatrix}}_{\mathbf{M}^0} \begin{bmatrix} \mathbf{c} \\ \mathbf{p} \end{bmatrix} - \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^{(0)} \end{bmatrix} = \mathbf{0}, \quad (43)$$

where

$$\mathbf{c} = \mathbf{u}_1^{(1)} \quad (44)$$

denoting the unknown beyond the contact forces. Matrix \mathbf{M}^0 of the obtained equation is positive semidefinite.⁶ To have positive quantities for unknowns, vector \mathbf{c} is produced as the difference of two positive vectors, i. e.

$$\begin{aligned} \mathbf{c} &= \mathbf{c}^+ - \mathbf{c}^-, \\ \mathbf{c}^+ &\geq \mathbf{0}, \quad \mathbf{c}^- \geq \mathbf{0} \end{aligned} \quad (45)$$

⁶ This statement is easy to prove. Since

$$(\mathbf{z}^T \mathbf{M}^0 \mathbf{z})^T = \mathbf{z}^T \mathbf{M}^{0,T} \mathbf{z},$$

it is:

$$\mathbf{z}^T (\mathbf{M}^0 + \mathbf{M}^{0,T}) \mathbf{z} = 2\mathbf{z}^T \mathbf{M}^0 \mathbf{z},$$

and

$$\mathbf{M}^0 + \mathbf{M}^{0,T} = 2 \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix},$$

a positive semidefinite matrix because of \mathbf{D} and \mathbf{H} .

and the first matrix equation in (43) is replaced by two matrix inequalities, resulting in

$$\underbrace{\begin{bmatrix} \mathbf{D} & -\mathbf{D} & \mathbf{G}^T \\ -\mathbf{D} & \mathbf{D} & -\mathbf{G}^T \\ -\mathbf{G} & \mathbf{G} & \mathbf{H} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \\ \mathbf{p} \end{bmatrix}}_{\mathbf{x}} - \underbrace{\begin{bmatrix} \mathbf{q} \\ -\mathbf{q} \\ \mathbf{t} \end{bmatrix}}_{\mathbf{d}} - \underbrace{\begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \\ \mathbf{y}^{(0)} \end{bmatrix}}_{\mathbf{y}} = \mathbf{0} \quad (46a)$$

or, more concisely:

$$\mathbf{M} \mathbf{x} - \mathbf{d} - \mathbf{y} = \mathbf{0}, \quad \mathbf{x}^T \mathbf{y} = \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}, \quad \mathbf{y} \geq \mathbf{0} \quad (46b-e)$$

where the positive semi-definiteness of \mathbf{M}^0 involves that of matrix \mathbf{M} .

From the Khun—Tucker theorems it is obvious that the programming problem belonging to (46) is:

$$\min \{ \psi(\mathbf{x}) = \mathbf{x}^T \mathbf{M} \mathbf{x} - \mathbf{x}^T \mathbf{d} \mid \mathbf{x} \geq \mathbf{0}, \quad -\mathbf{M} \mathbf{x} + \mathbf{d} \leq \mathbf{0} \} \quad (47)$$

easy to solve by the standard methods of quadratic programming [23]. Problem (47) can be considered as dual of (30).

In knowledge of \mathbf{x} and \mathbf{y} obtained by solving (47) and making use of (45), (44) yields $\mathbf{u}_1^{(1)}$, (35) yields $\mathbf{u}_1^{(1)}$, hence the generalized nodal displacement vector of body 1, and from (33) that of body 2, all these permitting the determination of deformation and stress states of the bodies.

1.2.4 Formulation of the quadratic programming problem where one body is rigid

Quite often, structures are encountered where one body can be considered as rigid compared to the other one.

1.2.4.1 Rigid body acted upon by a known system of forces

Assuming body 1 to be rigid and able to perform rigid-body motion. Let the vector be composed from the couple of vectors ($F_s^{(1)}$, $M_s^{(1)}$) reduced to the origin of the reference system of the known force system acting on this body denoted by

$$[\mathbf{f}_0^{(1),T} \mid \mathbf{m}_0^{(1),T}] = \mathbf{q}^{(1),T}, \quad (48)$$

where

$F_s^{(1)} \rightarrow \mathbf{f}_0^{(1)}$ being the resultant force, and
 $M_s^{(1)} \rightarrow \mathbf{m}_0^{(1)}$ the resultant moment. Applying (12b), potential energy of the system is:

$$\pi = \frac{1}{2} \mathbf{u}^{(2),T} \mathbf{K}^{(2)} \mathbf{u}^{(2)} - \mathbf{c}^{(T)} \mathbf{q}^{(1)} - \mathbf{u}^{(2),T} \mathbf{q}^{(2)}, \quad (49)$$

where

\mathbf{c} is the vector of the rigid-body motion of body 1 composed of its displacement along, and rotation about, the co-ordinate axes.

Lagrange function belonging to the contact problem can be written as:

$$L_3 = \pi + \int_{(\Omega)} p(w^{(1)} - w^{(2)} - h) dA. \quad (50)$$

Body 1 being rigid,

$$w^{(1)}(x) = \mathbf{A}_R^{(1)}(x) \mathbf{c} \quad (51)$$

$\mathbf{A}_R^{(1)}(x)$ being a matrix (row vector) depending on the body geometry, on the direction of unilateral relations.

Approximating the integral from the constraint in L_3 according to 1.2.2.2 and applying (20) to (22) and (51), we get:

$$\int_{(\Omega)} p(w^{(1)} - w^{(2)} - h) dA \cong \mathbf{p}^T [\mathbf{G}_R \mathbf{c} - \mathbf{A}^{(2)} \mathbf{u}^{(2)} - \mathbf{b}_R], \quad (52)$$

where $\mathbf{A}^{(2)}$, \mathbf{G}_R and \mathbf{b}_R are quantities obtained from (23b) and by omitting the last terms in (23c).

Repeating statements in item 1.2.3 yields functional belonging to the quadratic programming problem of the form:

$$\begin{aligned} L_4 = L_4(\mathbf{c}, \mathbf{u}^{(2)}, \mathbf{p}) = & \frac{1}{2} [\mathbf{c}^T \mid \mathbf{u}^{(2),T}] \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{u}^{(2)} \end{bmatrix} - \\ & - [\mathbf{c}^T \mid \mathbf{u}^{(2),T}] \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{q}^{(2)} \end{bmatrix} + \mathbf{p}^T ([\mathbf{G}_R \mid -\mathbf{A}^{(2)}] \begin{bmatrix} \mathbf{c} \\ \mathbf{u}^{(2)} \end{bmatrix} - \mathbf{b}_R) \end{aligned} \quad (53)$$

or, more concisely:

$$L_4 = \frac{1}{2} \mathbf{u}_R^T \mathbf{K}_R \mathbf{u}_R - \mathbf{u}_R^T \mathbf{q}_R + \mathbf{p}^T (\mathbf{A}_R \mathbf{u}_R - \mathbf{b}_R), \quad (54)$$

all quantities in (54) being obvious from a comparison with (53), and from the unilateral relation,

$$\mathbf{p} \geq \mathbf{0}, \quad -\mathbf{y}_R \equiv \mathbf{A}_R \mathbf{u}_R - \mathbf{b}_R \leq \mathbf{0}, \quad \mathbf{p}^T \mathbf{y}_R = 0.$$

The programming problem:

$$\min \left\{ \frac{1}{2} \mathbf{u}_R^T \mathbf{K}_R \mathbf{u}_R - \mathbf{u}_R^T \mathbf{q}_R \mid \mathbf{A}_R \mathbf{u}_R - \mathbf{b}_R \leq \mathbf{0} \right\}. \quad (55)$$

Dual of problem (55) is obtained by applying the Khun-Tucker theorems. Repeating statements in item 1.2.3.2 into the same sense, in conformity

with the Khun–Tucker theorem:

$$\frac{\partial L_4}{\partial \mathbf{c}} = -\mathbf{q}^{(1)} + \mathbf{G}_R^T \mathbf{p} = \mathbf{0}, \quad (56a)$$

$$\frac{\partial L_4}{\partial \mathbf{u}^{(2)}} = \mathbf{K}^{(2)} \mathbf{u}^{(2)} - \mathbf{q}^{(2)} - \mathbf{A}^{(2),T} \mathbf{p} = \mathbf{0} \quad (56b)$$

further

$$\frac{\partial L_4}{\partial \mathbf{p}} = \mathbf{G}_R \mathbf{c} - \mathbf{A}^{(2)} \mathbf{u}^{(2)} - \mathbf{b}_R \equiv -\mathbf{y}_R \leq \mathbf{0}, \quad (56c)$$

$$\mathbf{p}^T \frac{\partial L_4}{\partial \mathbf{p}} = \mathbf{p}^T \mathbf{y}_R = 0. \quad (56d)$$

Again assuming body 2 not performing any rigid-body motion, $\mathbf{u}^{(2)}$ can be expressed from (56b) and substituted into (56c):

$$-\mathbf{G}_R \mathbf{c} + \mathbf{H}^{(2)} \mathbf{p} - \mathbf{t}_R = \mathbf{y}_R \geq \mathbf{0}, \quad (57)$$

where

$$\mathbf{H}^{(2)} = \mathbf{A}^{(2)} [\mathbf{K}^{(2)}]^{-1} \mathbf{A}^{(2),T},$$

$$\mathbf{t}_R = -\mathbf{A}^{(2)} [\mathbf{K}^{(2)}]^{-1} \mathbf{q}^{(2)} - \mathbf{b}_R = -\mathbf{f}^{(2)} - \mathbf{b}_R.$$

Hypermatrix equation from (56a) and (57):

$$\begin{bmatrix} \mathbf{0} & \mathbf{G}_R^T \\ -\mathbf{G}_R & \mathbf{H}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{p} \end{bmatrix} - \begin{bmatrix} \mathbf{q}^{(1)} \\ \mathbf{t}_R \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{y}_R \end{bmatrix} = \mathbf{0}. \quad (58)$$

Again, producing vector \mathbf{c} as a difference of two positive vectors:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{G}_R^T \\ \mathbf{0} & \mathbf{0} & -\mathbf{G}_R^T \\ -\mathbf{G}^{(1)} & \mathbf{G}^{(1)} & \mathbf{H}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \\ \mathbf{p} \end{bmatrix} - \begin{bmatrix} \mathbf{q}^{(1)} \\ -\mathbf{q}^{(1)} \\ \mathbf{t}_R \end{bmatrix} - \begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \\ \mathbf{y}_R \end{bmatrix} = \mathbf{0}$$

$\underbrace{\quad}_{\mathbf{M}_R} \quad \underbrace{\quad}_{\mathbf{x}} \quad \underbrace{\quad}_{\mathbf{d}_R} \quad \underbrace{\quad}_{\mathbf{y}}$

or, concisely:

$$\mathbf{M}_R \mathbf{x} - \mathbf{d}_R - \mathbf{y} = \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x}^T \mathbf{y} = 0. \quad (59)$$

The relevant programming problem:

$$\min \{z(\mathbf{x}) = \mathbf{x}^T \mathbf{M}_R \mathbf{x} - \mathbf{x}^T \mathbf{d}_R \mid \mathbf{x} \geq \mathbf{0}, \quad -\mathbf{M}_R \mathbf{x} + \mathbf{d}_R \leq \mathbf{0}\}. \quad (60)$$

Obviously, the final programming problem is the same, either both contacting bodies are elastic or one of them is rigid. Of course, equilibrium equations (58) and (43) are essentially different by physical purport: one referring to the body as a whole, the other to given nodes for the support of the “primary structure” ($\det \mathbf{K}_{22}^{(1)} \neq 0$).

1.2.4.2 Known displacement of a rigid body

In the case where rigid-body displacement \mathbf{c} of body 1 is a given value, no equilibrium equations need be written for body 1, the contact problem can be set up by means of the geometry equation/inequality expressing the contact separation.

Omitting deduction:

$$\mathbf{H}^{(2)} \mathbf{p} - (\mathbf{G}_R \mathbf{c} + \mathbf{t}_R) \equiv \mathbf{y}_R \geq \mathbf{0}, \quad \mathbf{p}^T \mathbf{y}_R = 0, \quad (61a)$$

where

$$\begin{aligned} \mathbf{t}_R = & -\mathbf{A}^{(2)} [\mathbf{K}^{(2)}]^{-1} \mathbf{q}^{(2)} - \left\{ \int_{(\Omega)} \mathbf{P}(x) \mathbf{L}(x) dA \mathbf{h} + \right. \\ & \left. + \int_{(\Omega)} \mathbf{P}(x) \dot{\mathbf{A}}^{(2)}(x) dA \dot{\mathbf{u}}^{(2)} \right\} \end{aligned} \quad (61b)$$

$\mathbf{H}^{(2)}$ being a positive definite matrix, the programming problem is:

$$\min \left\{ A(\mathbf{p}) = \frac{1}{2} \mathbf{p}^T \mathbf{H}^{(2)} \mathbf{p} - \mathbf{p}^T (\mathbf{G}_R \mathbf{c} + \mathbf{t}_R) \mid \mathbf{p} \geq \mathbf{0} \right\}. \quad (62)$$

1.2.5 Another possibility for solving the contact problem

Let us remark that if both bodies are elastic, one construction of the problem may be similar to (58), except for $\mathbf{H}^{(2)}$ being replaced by the sum of the influence function matrix $\mathbf{H}^{(1)}$ based on the "primary structure" 1 and of $\mathbf{H}^{(2)}$, \mathbf{t}_R is modified to the same sense, replaced by

$$\mathbf{t} = \mathbf{f}^{(1)} - \mathbf{f}^{(2)} - \mathbf{b}$$

terms delivered by (40), (41) and (23). Interpretation of vector \mathbf{c} for a plane structure is to be seen in Fig. 3.

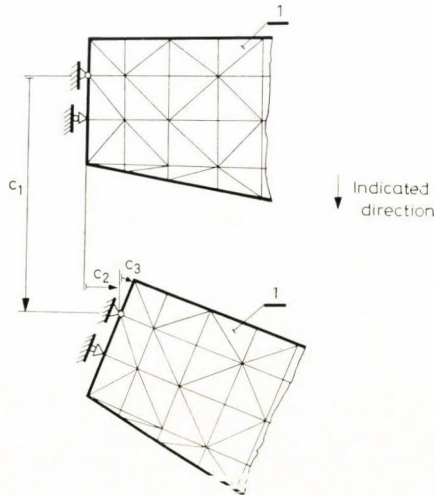


Fig. 3. Body 1 of the plane structure may perform rigid-body motion. Interpretation of vector \mathbf{c} : $\mathbf{c}^T = [c_1, c_2, c_3]$

Such a setup of the problem is considered in [1], [2] and [9], the solution being obtained by a cyclic iteration according to the method of gradients.

Obviously, such a setup of the problem cannot be considered as the strict dual of programming problem (30), (47) being the dual of problem (30).

1.3 Comments on practical computations

In actual applications of the described method, setup of influence matrices (41a, b) involves the inversion of matrices $\mathbf{K}^{(t)}$. Complicated body designs often call for a high number of elements to be assumed, preventing the $\mathbf{K}^{(t)}$ for the entire structure to be kept in the active storage unit of the computer. In this case, influence matrices $\mathbf{H}^{(t)}$ are advisably produced as follows:

1. In case of body 1 performing rigid-body motion, a "primary structure" with a kind of support has to be assumed, so that the relevant $\mathbf{K}_{22}^{(1)}$ will not degenerate any longer, and supports have to be located in a domain where matrix $\mathbf{A}_1^{(1)}$ is absolutely zero (see in (23), (34); see e. g. Fig. 4).

2. Respective J-th columns of matrices $\mathbf{H}^{(t)}$, $t = 1, 2$, will be delivered by vectors obtained by multiplying from the left the displacement vectors

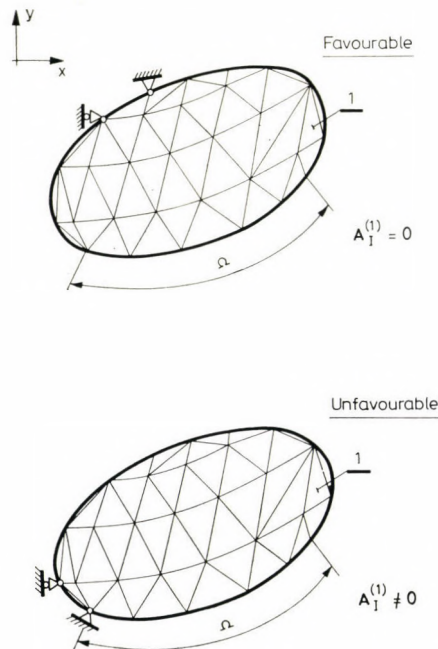


Fig. 4. Assumption of basic structure for a plane structure if co-ordinates of the generalized nodal displacement vector are displacements

at nodes $\mathbf{u}_J^{(1)}$ and $\mathbf{u}_J^{(2)}$, arrived at by solving algebraic equations

$$\begin{aligned}\mathbf{K}_{22}^{(1)} \mathbf{u}_J^{(1)} &= \mathbf{A}_{11}^{(1),T} \mathbf{p}, \\ \mathbf{K}^{(2)} \mathbf{u}_J^{(2)} &= \mathbf{A}^{(2),T} \mathbf{p}\end{aligned}$$

related to loads for unit values of vector \mathbf{p} in (22) at different points J .

3. Knowing $\mathbf{K}_{11}^{(1)}$, $\mathbf{K}_{12}^{(1)} = \mathbf{K}_{21}^{(1),T}$ obtained in producing the "primary structure" of structure 1, matrices \mathbf{D} and \mathbf{G} can easily be calculated.

4. Nodes in domain Ω are advisably consecutively numbered; thereby matrix $\mathbf{A}^{(i)}$ will become quasidiagonal, greatly simplifying calculations.

Conclusions

The contact problem was seen to be constructed on the Lagrange variation principle, and treated as a quadratic programming problem owing to unilateral relations. The kinematically possible displacement field is approximated by the finite element method, and the condition of contact/separation is checked in a predetermined (indicated) direction at a finite discrete number of points according to item 1.2.2.1 and by comparing integral values for small finite sections according to item 1.2.2.2 (see (23) to (25)).

This latter procedure is justified anyway in case of a field of displacement vectors approximated by a higher than linear polynomial element-wise, and it is especially adequate for thin shells and plates. Use of elements of higher degrees of freedom leads to a decrease in the size of the final algebraic equation system, permitting an important time saving, and besides, yielding a true picture of the stress state prevailing on the contact domain.

Appendix

Essentials of the theory of quadratic programming.

A quadratic programming problem I is understood as minimizing the strictly quasi-convex quadratic function⁷:

$$l(\mathbf{x}) = \mathbf{x}^T \mathbf{C} \mathbf{x} - \mathbf{c}^T \mathbf{x} + \gamma, \quad \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x}^T \mathbf{C} \mathbf{x} \geq 0, \quad (\text{A. 1})$$

subject to the constraints of equation-inequality

$$\mathbf{A} \mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0} \quad (\text{A. 2})$$

$$\text{I. min } \{ \mathbf{x}^T \mathbf{C} \mathbf{x} - \mathbf{c}^T \mathbf{x} + \gamma \mid \mathbf{x} \geq \mathbf{0}, \mathbf{A} \mathbf{x} \leq \mathbf{b} \} \quad (\text{A. 3a})$$

⁷ A function $l(\mathbf{x})$ is quasi-convex if for any $0 < \lambda < 1$ of the convex domain $\mathbf{x} \in D$ and for conditions \mathbf{x}^1 and $\mathbf{x}^2 \in D$ the inequality

$$l(\lambda \mathbf{x}^1 + (1 - \lambda) \mathbf{x}^2) \leq \lambda l(\mathbf{x}^1) + (1 - \lambda) l(\mathbf{x}^2)$$

holds; this is invariably the case if \mathbf{C} is a positive semidefinite matrix.

Besides, depending on the constraints, also problems II and III are spoken about, namely:

$$\text{II. } \min \{ \mathbf{x}^T \mathbf{C} \mathbf{x} - \mathbf{c}^T \mathbf{x} + \gamma \mid \mathbf{A} \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \} \quad (\text{A. 3b})$$

and

$$\text{III. } \min \{ \mathbf{x}^T \mathbf{C} \mathbf{x} - \mathbf{c}^T \mathbf{x} + \gamma \mid \mathbf{A} \mathbf{x} \leq \mathbf{b} \} \quad (\text{A. 3c})$$

Problems II and III can be transformed to problem I; II by means of inequalities $\mathbf{A} \mathbf{x} \leq \mathbf{b}$ and $-\mathbf{A} \mathbf{x} \leq -\mathbf{b}$; III requiring a variable transformation $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$, $\mathbf{x}^+ \geq \mathbf{0}$, $\mathbf{x}^- \geq \mathbf{0}$, $\tilde{\mathbf{x}}^T = [\mathbf{x}^{+,T} \mathbf{x}^{-,T}] \geq \mathbf{0}$.

The conditional extreme value calculation is generalized by the Khun-Tucker theorem for the case where the constraints not only contain equations but also inequalities. According to this theorem, the Lagrange function belonging to (A. 3a):

$$L = \mathbf{x}^T \mathbf{C} \mathbf{x} - \mathbf{c}^T \mathbf{x} + \gamma + \mathbf{u}^T (\mathbf{A} \mathbf{x} - \mathbf{b}) \quad (\text{A. 4})$$

has a saddle point for the optimum solution for \mathbf{x}^0 of the minimization problem (A. 3a), hence there is vector \mathbf{u}^0 with an inequality relationship

$$L(\mathbf{x}, \mathbf{u}^0) \geq L(\mathbf{x}^0, \mathbf{u}^0) \geq L(\mathbf{x}^0, \mathbf{u}) \quad (\text{A. 5})$$

for any $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{u} \geq \mathbf{0}$. This fulfils of so-called local conditions

$$\mathbf{v} = \frac{\partial L}{\partial \mathbf{x}} \Big|_{(\mathbf{x}^0, \mathbf{u}^0) \geq \mathbf{0}, \mathbf{x}^0, T} \frac{\partial L}{\partial \mathbf{x}} \Big|_{(\mathbf{x}^0, \mathbf{u}^0)} = \mathbf{0}, \quad (\text{A. 6a, b})$$

$$\mathbf{y} = -\frac{\partial L}{\partial \mathbf{u}} \Big|_{(\mathbf{x}^0, \mathbf{u}^0) \geq \mathbf{0}, \mathbf{u}^0, T} \frac{\partial L}{\partial \mathbf{u}} \Big|_{(\mathbf{x}^0, \mathbf{u}^0)} = \mathbf{0} \quad (\text{A. 7a, b})$$

as necessary and sufficient conditions. $\partial L / \partial \mathbf{x}$ being a column vector with components obtained by taking the partial derivative of function $L(\mathbf{x}, \mathbf{u})$ with respect to components of \mathbf{x} ; $\partial L / \partial \mathbf{u}$ being a similarly interpreted vector. Lacking a non-negativity requirement for \mathbf{x} , (A. 6a) is replaced by relationship [23]:

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{0}. \quad (\text{A. 8})$$

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Anwendung der Verschiebungsmethode der finiten Elemente zur Lösung von Problemen der elastischen Berührung. Die Klärung des Berührungsproblems elastischer Kontinua bei beliebiger Belastung und Fläche ist sehr umständlich, da die Berührungsbereiche im vorherein nicht bekannt sind. Im Beitrag wird das Kontinuum durch eine Menge von Elementen mit endlichen Freiheitsgraden ersetzt und auf das so erhaltene elastische System wird nach dem Prinzip des Minimums der Potentialenergie die Lösung der Aufgabe aufgebaut, die sich wegen der einseitigen Verbindungen zwischen den Körpern (die Kontaktkraft kann nur gegen das Innere des Körpers gerichtet sein und wird dann als positiv betrachtet) als quadratische Programmierungsaufgabe der mathematischen Programmierung behandeln läßt. Unter Anwendung der Kuhn—Tuckerschen Bedingungen ist auch die zu dem obigen Primalproblem gehörende Dualaufgabe geklärt, die es bei konkreten Berechnungen vorteilhafter aufzustellen und zu lösen ist, als die ursprüngliche Aufgabe. Es wird vorausgesetzt, daß die Reibung und Haftung zwischen den Körpern vernachlässigt werden können und daß die Verschiebungen, Formänderungen gering sind.

Применение метода конечных элементов для решения контактных задач. Решение контактных задач для сплошных сред при произвольных нагрузках и конфигурациях является очень сложным, так как заранее неизвестны области контакта. В данной работе сплошная среда замещается множеством элементов с конечной степенью свободы. На полученную таким образом упругую систему базируется — на основе минимума потенциальной энергии — решение задачи, которая вследствие односторонней связи между телами (контактное давление может быть направлено только в тело, и в этом случае оно считается положительным) в конечном счете сведена к решению задачи квадратического программирования. При использовании условий Куна-Таккера выяснена дуальная задача, относящаяся к выше упомянутой примальной задаче. Постановка и решение дуальной задачи при конкретных вычислениях кажется более выгодной, чем решение примальной задачи. Предполагается, что трением и сцеплением между телами можно пренебречь, далее перемещение и деформация являются малыми.

