# SECOND-ORDER ANALYSIS OF COMPOSITE BEAM-COLUMNS WITH INTERLAYER SLIP

***Ákos József Lengyel*[1], *István Ecsedi*[2]**
[1]Assistant Lecturer, [2]Emeritus Professor of Mechanics
*[1,2]Institute of Applied Mechanics, University of Miskolc, Miskolc-Egyetemváros, H-3515 Hungary*
*e-mail: [1]mechlen@uni-miskolc.hu, [2]mechecs@uni-miskolc.hu*

**Abstract**
Exact second-order analysis of two-layer composite beam-columns with weak shear connection which are subjected to transverse axial loads is presented. Closed-form solutions for the deflection, interlayer slip, bending moment and shear force are derived for simply supported beam-columns. Results of the present paper are compared with the solutions obtained by a different method as which is used in this paper.

## 1. INTRODUCTION

The present paper deals with the solution of a static problem of two-layer composite beam-columns with imperfect shear connection. The considered simply supported beam-column and its load are shown in Fig. 1. The beam-column carries the transverse load and constant axial load. The cross section of the composite beam is also shown in Fig. 1. The modulus of elasticity of beam component $i$ is $E_i$ and its cross section is $A_i$ $(i=1,2)$. The centre of the cross-section $A_i$ is denoted by $C_i$ $(i=1,2)$. The $E$-weighted centre of the composite cross section is $C$. In the presented numerical example which is borrowed from paper [1] the point $C$ is on the common boundary of $A_1$ and $A_2$ as shown in Fig. 1. The length of the beam-column is denoted by $L$. The origin $O$ of the rectangular Cartesian coordinate system $Oxyz$ is the $E$-weighted centre of the left end cross section $A=A_1\cup A_2$, so that the axis $z$ is the $E$-weighted centreline of the considered two-layer composite beam-column with flexible shear connection. Denote the beam-column component $i$ is $B_i$ $(i=1,2)$. A point $Q$ in $B=B_1\cup B_2$ is illustrated by the position vector $\overrightarrow{OQ}=\mathbf{r}=x\mathbf{e}_x+y\mathbf{e}_y+z\mathbf{e}_z$, where $\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z$ are the unit vectors of the coordinate system $Oxyz$. It is known [2] that the position of the $E$-weighted centre $C$ is obtained from next equations (Fig. 1)

$$c_1=\left|\overrightarrow{CC_1}\right|=\frac{A_2E_2}{\langle AE\rangle}c,\quad c_2=\left|\overrightarrow{CC_2}\right|=\frac{A_1E_1}{\langle AE\rangle}c,\quad c=\left|\overrightarrow{C_1C_2}\right|=h_1+h_2,\quad \langle AE\rangle=A_1E_1+A_2E_2. \quad (1)$$

The common boundary of the beam-column component $B_1$ and $B_2$ is determined by $y=0$, $|x|\le 0.5b_2$ (Fig. 1). The applied axial force acts on beam-column component $B_i$ is denoted by $P_i$ $(i=1,2)$. The point of application of $P_i$ is the point $C_i$ $(i=1,2)$.

The magnitude of $P_1$ and $P_2$ are such that the point of application of the resultant axial force $P = P_1 + P_2$ is the $E$-weighted centre of the composite cross section. From this fact it follows that (Fig. 2)

$$P_1 = \frac{c_2}{c} P, \quad P_2 = \frac{c_1}{c} P. \tag{2}$$

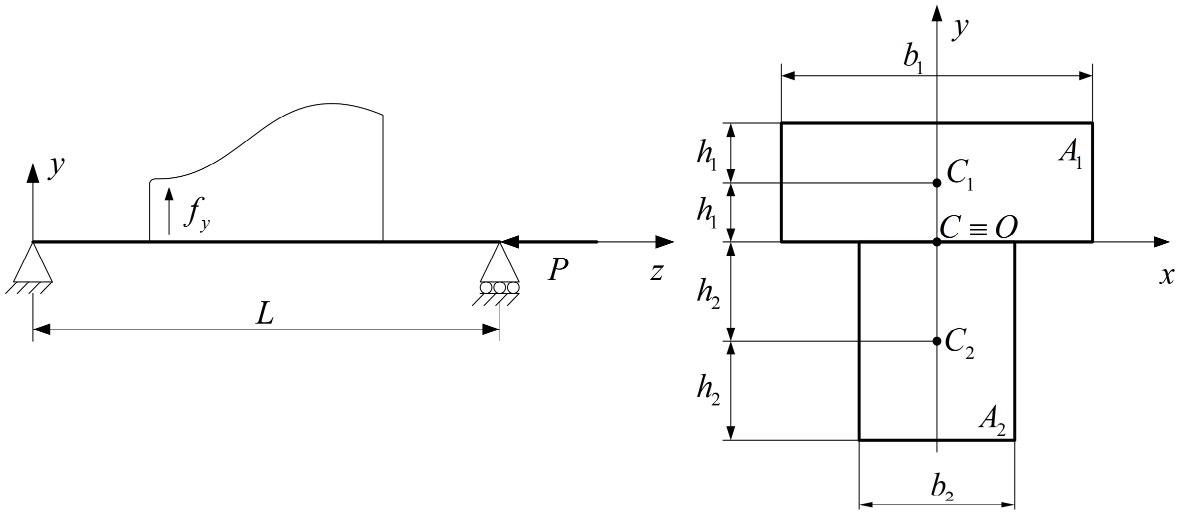



Figure 1. Two-layer composite beam column and its cross section.

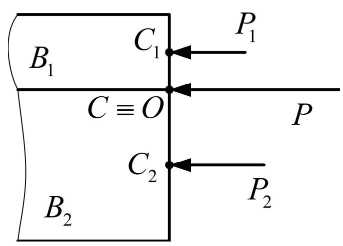



Figure 2. Illustration of the applied axial load.

## 2. GOVERNING EQUATION

According to the Euler-Bernoulli beam theory, which is valid for each homogeneous beam-column components, the deformed configuration in presence of constant axial load is described by the next displacement field

$$u = 0, \quad v = v(z), \quad (x, y, z) \in B_1 \cup B_2, \tag{3}$$

$$w(y,z) = w_1(z) - y\frac{\mathrm{d}v}{\mathrm{d}z} - \frac{P_1}{A_1 E_1} z, \quad (x, y, z) \in B_1, \tag{4}$$

$$w(y,z) = w_2(z) - y\frac{\mathrm{d}v}{\mathrm{d}z} - \frac{P_2}{A_2 E_2} z, \quad (x, y, z) \in B_2. \tag{5}$$

In Eqs. (3-5) $u$ is the displacement in direction of $\mathbf{e}_x$, $v$ is the displacement in direction of $\mathbf{e}_y$ and $w$ is the axial displacement (Fig. 1). On the common boundary of $B_1$ and $B_2$ the axial displacement may have jump which is called the interlayer slip

$$s(z) = w_1(z) - w_2(z) - z\left(\frac{P_1}{A_1E_1} - \frac{P_2}{A_2E_2}\right). \tag{6}$$

A simple computation shows that according to Eqs. (1) and (2)

$$\frac{P_1}{A_1E_1} - \frac{P_2}{A_2E_2} = 0, \tag{7}$$

that is

$$s(z) = w_1(z) - w_2(z). \tag{8}$$

Application of the strain-displacement relationships of the linearized theory of elasticity gives [3,4]

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0, \quad (x,y,z) \in B, \tag{9}$$

$$\varepsilon_z = \frac{\mathrm{d}w_1}{\mathrm{d}z} - y\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - \frac{P_1}{A_1E_1}, \quad (x,y,z) \in B_1, \tag{10}$$

$$\varepsilon_z = \frac{\mathrm{d}w_2}{\mathrm{d}z} - y\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - \frac{P_2}{A_2E_2}, \quad (x,y,z) \in B_2. \tag{11}$$

From the Hooke's law for the normal stress field the next result can be derived

$$\sigma_z = E_1\left(\frac{\mathrm{d}w_1}{\mathrm{d}z} - y\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\right) - \frac{P_1}{A_1}, \quad (x,y,z) \in B_1, \tag{12}$$

$$\sigma_z = E_2\left(\frac{\mathrm{d}w_2}{\mathrm{d}z} - y\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\right) - \frac{P_2}{A_2}, \quad (x,y,z) \in B_2. \tag{13}$$

The following sub-section forces and moments are introduced

$$N_1 = \int_{A_1} \sigma_z \mathrm{d}A = n_1 - P_1, \quad N_2 = \int_{A_2} \sigma_z \mathrm{d}A = n_2 - P_2, \tag{14}$$

where

$$n_1 = E_1A_1\left(\frac{\mathrm{d}w_1}{\mathrm{d}z} - c_1\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\right), \quad n_2 = E_2A_2\left(\frac{\mathrm{d}w_2}{\mathrm{d}z} + c_2\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\right). \tag{15}$$

$$M_1 = \int_{A_1} y\sigma_z \mathrm{d}A = c_1E_1A_1\frac{\mathrm{d}w_1}{\mathrm{d}z} - E_1I_1\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - c_1P_1, \tag{16}$$

$$M_2 = \int_{A_2} y\sigma_z \mathrm{d}A = -c_2E_2A_2\frac{\mathrm{d}w_2}{\mathrm{d}z} - E_2I_2\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + c_2P_2. \tag{17}$$

Here,

$$I_i = \int_{A_i} y^2 \mathrm{d}A, \quad (i=1,2). \tag{18}$$

It is evident, that

$$N_1 + N_2 = -P, \tag{19}$$

that is

$$n_1 + n_2 = 0. \tag{20}$$

Equation (20) can be written in the form

$$E_1 A_1 \left( \frac{\mathrm{d}w_1}{\mathrm{d}z} - c_1 \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} \right) + E_2 A_2 \left( \frac{\mathrm{d}w_2}{\mathrm{d}z} + c_2 \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} \right) = 0. \tag{21}$$

From the definition of the interlayer slip we have

$$\frac{\mathrm{d}w_1}{\mathrm{d}z} - \frac{\mathrm{d}w_2}{\mathrm{d}z} = \frac{\mathrm{d}s}{\mathrm{d}z}. \tag{22}$$

Eqs. (21), (22) form a system of linear equations for $\frac{\mathrm{d}w_1}{\mathrm{d}z}$ and $\frac{\mathrm{d}w_2}{\mathrm{d}z}$. A simple computation yields that

$$-c_1 E_1 A_1 + c_2 E_2 A_2 = 0. \tag{23}$$

Combination of Eqs. (21), (22) with Eq. (23) gives

$$\frac{\mathrm{d}w_1}{\mathrm{d}z} = \frac{E_2 A_2}{\langle AE \rangle} \frac{\mathrm{d}s}{\mathrm{d}z} = \frac{c_1}{c} \frac{\mathrm{d}s}{\mathrm{d}z}, \quad \frac{\mathrm{d}w_2}{\mathrm{d}z} = -\frac{E_1 A_1}{\langle AE \rangle} \frac{\mathrm{d}s}{\mathrm{d}z} = -\frac{c_2}{c} \frac{\mathrm{d}s}{\mathrm{d}z}. \tag{24}$$

Inserting these results into the expression of $N_1$ we have

$$N_1 = \langle AE \rangle_{-1} \left( \frac{\mathrm{d}s}{\mathrm{d}z} - c \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} \right) - P_1 = n_1(z) - P_1, \tag{25}$$

where

$$\langle AE \rangle_{-1} = \frac{A_1 E_1 A_2 E_2}{\langle AE \rangle}. \tag{26}$$

Application of the condition of equilibrium for the forces acting in axial direction of beam-column component $B_1$ the following equation can be obtained [2]

$$\frac{\mathrm{d}N_1}{\mathrm{d}z} - T = \langle AE \rangle_{-1} \left( \frac{\mathrm{d}^2 s}{\mathrm{d}z^2} - c \frac{\mathrm{d}^3 v}{\mathrm{d}z^3} \right) - ks = 0. \tag{27}$$

In Eq. (27) $T = ks$ is the interlayer shear force and $k$ is the slip modulus [1,2]. The total bending moment is as follows (Fig. 3)

$$M = M_1 + M_2 = c\langle AE\rangle_{-1}\frac{\mathrm{d}s}{\mathrm{d}z} - \{IE\}\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}, \tag{28}$$

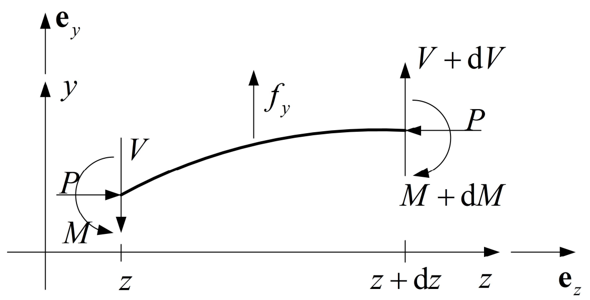



Figure 3. Beam element with its loads.

where $\{IE\} = I_1E_1 + I_2E_2$. According to Fig. 3 we have the following equilibrium equations

$$\frac{\mathrm{d}V}{\mathrm{d}z} + f_y = 0, \quad \frac{\mathrm{d}M}{\mathrm{d}z} - V - P\frac{\mathrm{d}v}{\mathrm{d}z} = 0. \qquad (29)_{1,2}$$

In Eqs. (29) $f_y = f_y(z)$ is the applied distributed force in direction of $y$ axis, $V = V(z)$ is the shear force (Fig. 3). From Eqs. (29) we obtain

$$\frac{\mathrm{d}^2 M}{\mathrm{d}z^2} - P\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + f_y = 0. \tag{30}$$

Substituting the expression $M = M(z)$ given by Eq. (28) into Eq. (30) provides the next result

$$c\langle AE\rangle_{-1}\frac{\mathrm{d}^3 s}{\mathrm{d}z^3} - \{IE\}\frac{\mathrm{d}^4 v}{\mathrm{d}z^4} - P\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + f_y = 0. \tag{31}$$

The determination of deflection and slip functions is based on the solution of system of equations (27), (31). Here, we note that the first-order analysis is based on the next equilibrium equation

$$\frac{\mathrm{d}M}{\mathrm{d}z} - V = 0. \tag{32}$$

## 3. SOLUTION FOR SIMPLY SUPPORTED BEAM-COLUMNS

Figure 1 shows a simply supported beam-column with distributed transverse load. The boundary conditions for simply supported beam-column are as follows

$$v(0)=0,\quad M(0)=0,\quad n(0)=0,\quad v(L)=0,\quad M(L)=0,\quad n(L)=0. \tag{33}$$

We will use the Fourier' series representation of the applied load in which means

$$f_y(z)=\sum_{j=1}^{\infty} f_j \sin\frac{j\pi}{L}z. \tag{34}$$

We look for the solution of system of equations (27), (31) in the next form

$$v(z)=\sum_{j=1}^{\infty} V_j \sin\frac{j\pi}{L}z,\quad s(z)=\sum_{j=1}^{\infty} S_j \cos\frac{j\pi}{L}z. \tag{35}$$

A simple computation gives the next results for $n=n(z)$ and $M=M(z)$

$$n(z)=-\langle AE\rangle_{-1}\sum_{j=1}^{\infty}\left[\frac{j\pi}{L}S_j+c\left(\frac{j\pi}{L}\right)^2 V_j\right]\sin\frac{j\pi}{L}z, \tag{36}$$

$$M(z)=\sum_{j=1}^{\infty}\left[-c\frac{j\pi}{L}\langle AE\rangle_{-1}S_j+\left(\frac{j\pi}{L}\right)^2\{IE\}V_j\right]\sin\frac{j\pi}{L}z. \tag{37}$$

It is evident functions provided by Eqs. (35-37) satisfy the boundary conditions formulated in Eqs. (33). From Eqs. (27), (31) and Eqs. (35) it follows

$$V_j=\frac{f_j}{v_j},\quad S_j=\frac{c\left(\frac{j\pi}{L}\right)^3}{\left(\frac{j\pi}{L}\right)^2+\frac{k}{\langle AE\rangle_{-1}}}V_j,\quad (j=1,2,...), \tag{38}$$

$$v_j=-\left(\frac{j\pi}{L}\right)^2\left[\frac{c^2\langle AE\rangle_{-1}\left(\frac{j\pi}{L}\right)^4}{\left(\frac{j\pi}{L}\right)^2+\frac{k}{\langle AE\rangle_{-1}}}-\left(\frac{j\pi}{L}\right)^2\{IE\}+P\right],\quad (j=1,2,...). \tag{39}$$

4. NUMERICAL EXAMPLE

This example is taken from paper by Girhammar and Gopu [1]. The simply supported beam is shown in Fig. 1. The applied load is uniform distirbuted load acting on the whole length of the beam-column, that is $f_y(z)=-f$, $(0<z<L)$. The cross section of the considered two-layer composite beam-column is also given in Fig. 1. The next numerical data will be used (Fig. 1): $h_1=0.025\,[\mathrm{m}]$, $h_2=0.075\,[\mathrm{m}]$, $b_1=0.3\,[\mathrm{m}]$, $b_2=0.05\,[\mathrm{m}]$, $E_1=2\times10^{10}$ [Pa], $E_2=8\times10^{9}$ [Pa], $k=50\times10^{6}$ [Pa], $P=5\times10^{4}$ [N], $f=1000$ [N/m], $L=4$ [m]. In the following two cases are going to

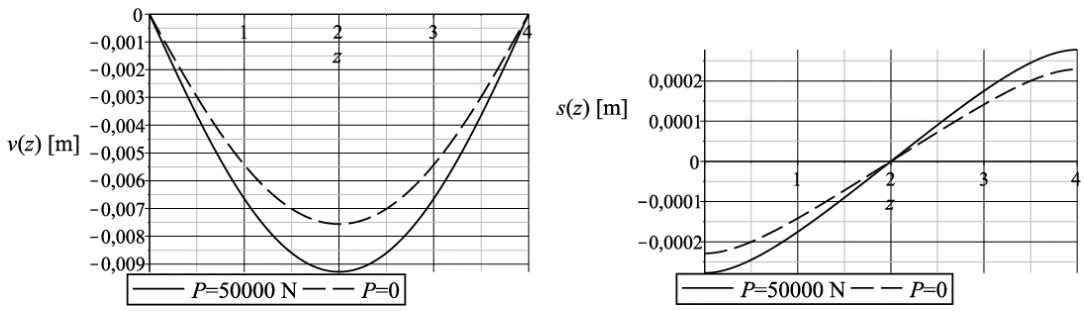
be considered. The first case involves the first order analysis ($P = 0$ [N]) while the second case implies the second order analysis ($P = 5\times10^4$ [N]). The other data are the same for both cases. In Figure 4a and 4b the graphs of the deflection functions and the slip functions are shown, respectively. The graphs of the bending moment fuctions and the shear force functions are illustrated in Figs. 5a and 5b, respectively. The shear force functions for the second-order and for the first-order analyses are the same as shown in Fig. 11. The problem for $P = 5\times10^4$ [N] was analysed in paper by Girhammar and Gopu [1]. The comparison of results obtained in [1] and in the present paper is given in Table 1. We note that the shear force $T(L)$ in all cases can be gained from equilibrium equation of statics, exact value of that $T(L) = 2000$ [N]. In the case of first-order analysis the exact value of $|M(L/2)| = 2000$ [Nm] which can be derived by the application of statics. In Table 1, $M(L/2)$ is computed from Eq. (26) and $V = V(z)$ is obtained from next equations:

for second-order analysis

$$V(z) = c\langle AE\rangle_{-1}\frac{d^2 s}{dz^2} - \{IE\}\frac{d^3 v}{dz^3} - P\frac{dv}{dz}, \tag{40}$$

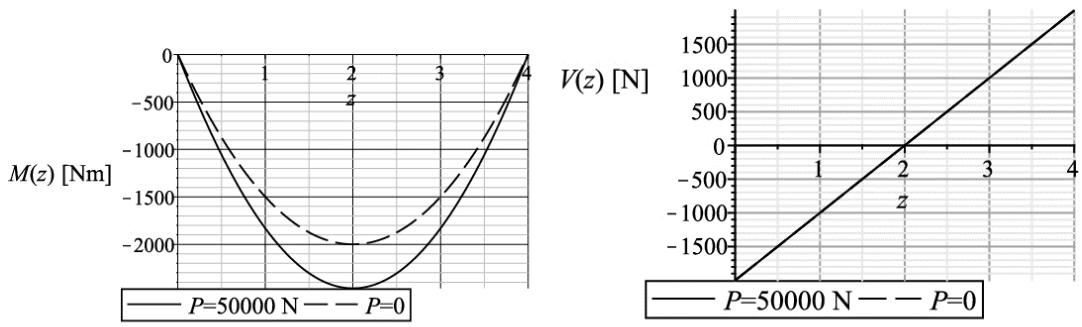
for first-order analysis

$$V(z) = c\langle AE\rangle_{-1}\frac{d^2 s_1}{dz^2} - \{IE\}\frac{d^3 v_1}{dz^3}. \tag{41}$$




a) b)

Figure 4. The graphs of deflection and slip functions.




a) b)

Figure 5. The plots of the bending moment and the shear force functions.

Table 1. Comparison of results of second-order analysis and first-order analysis.

| | second-order analysis | | first-order analysis |
|---|---|---|---|
| | paper [1] $P = 50000$ N | this paper $P = 50000$ N | |
| $\lvert v(L/2)\rvert$ [m] | 0.009276 | 0.009280398129 | 0.007559897176 |
| $\lvert s(L)\rvert$ [m] | 0.0002775 | 0.000277736064 | 0.000228879711 |
| $\lvert M(L/2)\rvert$ [Nm] | 2461.3 | 2464.019876 | 2000 |
| $T(L)$ [N] | - | 1995.947202 | 1999.189428 |

In Eq. (41) $v_1 = v_1(z)$ and $s_1 = s_1(z)$ denote the deflection and slip functions, respectively, which are obtained by application of the governing equations of the first-order analysis.

## 5. CONCLUSIONS

Analytical solutions are developed for the deflection, slip and bending moment for two-layer composite beam-columns with imperfect shear connection. The considered simply supported composite beam-column is subjected to transverse and axial load. The results of the first-order and second-order analyses are compared. The obtained results illustrate the second-order effect to the displacement, slip and bending moment functions.

## ACKNOWLEDGEMENTS

This research was supported by the National Research, Development and Innovation Office – NKFIH, K115701.

This research was also carried out as part of the EFOP-3.6.1-16-2016-00011 Younger and Renewing University – Innovative Knowledge City – institutional development of the University of Miskolc aiming at intelligent specialisation" project implemented in the framework of the Szechenyi 2020 program. The realization of this project is supported by the European Union, co-financed by the European Social Fund.

## REFERENCES

[1] GIRHAMMAR, U. A., GOPU, V. K. A.: **Composite beam-columns with interlayer slip – exact analysis.** Journal of Structural Engineering 119(4), 1993, 1265-1282.

[2] ECSEDI, I., BAKSA, A.: **Static analysis of composite beams with weak shear connection.** Applied Mathematical Modelling 35(4), 2011, 1739-1750.

[3] SOKOLNIKOFF, I. S.: **Mathematical Theory of Elasticity.** McGraw-Hill, New York, 1970, 2nd Edition

[4] SLAUGHTER W. S.: **The linearized theory of elasticity.** Birkhäuser, Basel, 2002.