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TORSION OF CYLINDRICALLY ORTHOTROPIC COMPOSITE BAR WITH CROSS SECTION OF A SECTOR OF SOLID CIRCLE

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Abstract

This paper deals with the Saint-Venant torsion of elastic cylindrically orthotropic composite bar whose cross section is a sector of a solid circle. An analytical method is presented to obtain the Prandtl's stress function and torsion function of the compound cross section made of two different cylindrically orthotropic material.

1 INTRODUCTION

The Saint-Venant torsion of anisotropic linearly elastic bars has been the subject of several studies from both theoretical and numerical viewpoints. Books by Lekhnitskii [1,2], Sarkisyan [3,4], Rand and Rovenski [5] give the detailed analyses of Saint-Venant torsion of anisotropic and orthotropic bars. In these books both the Prandtl's stress function and torsion function formulations are used. Arutyunyan and Abramyan give a detailed analysis of the uniform torsion for bars whose cross sections consist of different isotropic homogeneous elastic materials [6]. The warping properties of twisted funcionally graded anisotropic linear elastic bars are treated in a paper by Horgan [7]. Ely and Zienkiewicz give a finite difference solution for Saint-Venant torsion of a composite rectangular cross section made of two different isotropic homogeneous materials [8]. This problem is also solved in [9] with a variational formulation of the torsion problem for compound cross section. Bevilacqua et al. give an analytical solution for the Saint-Venant torsion of composite rectangular cross section [9]. Papers [10–14] present the formulation of uniform torsion and some solutions for cylindrically anisotropic elastic bars. The object of this paper is the Saint-Venant torsion of cylindrically orthotropic composite cross section made of two different material. The shape of the considered cross section is a sector of solid circle as shown in Fig. 1. The cylindrically anisotropic materials having linear elastic stress-strain relation can possess a unique type of coupling between the radial and circumferential directions. The cylindrical orthotropy is a lower level of the cylindrical anisotropy furthermore it has a weaker coupling between the radial and circumferential directions as in the case of cylindrical anisotropy.

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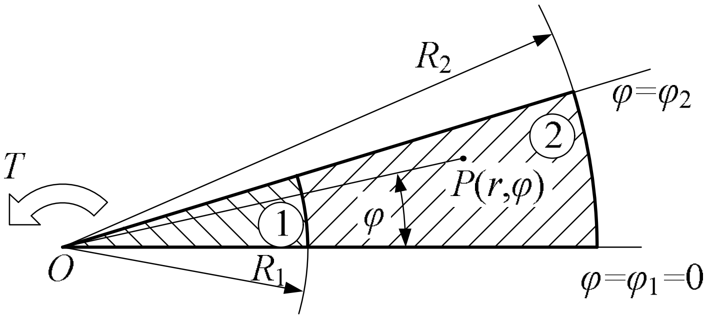

Fig.1
Composite sector of solid circle

2 GOVERNING EQUATIONS

We consider the Saint-Venant torsion of composite cylindrically orthotropic linear elastic bar whose cross section is shown in Fig. 1. The formulation of the torsion problem is given in cylindrical polar coordinates r,φ (Fig. 1). The cross section of the considered torsion problem is denoted by A and its boundary curve by ∂A, where $A = A_1 \cup A_2$ and $\partial A = \partial A_{11} \cup \partial A_{12} \cup \partial A_{21} \cup \partial A_{22} \cup \partial A_3$ (Fig. 1),

$$\begin{gathered} A_1 = \{(r,\varphi) \mid 0 \le r \le R_1, \varphi_1 \le \varphi \le \varphi_2\}, \quad A_2 = \{(r,\varphi) \mid R_1 \le r \le R_2, \varphi_1 \le \varphi \le \varphi_2\}, \\ \partial A_{11} = \{(r,\varphi) \mid 0 \le r \le R_1, \varphi = \varphi_1\}, \quad \partial A_{12} = \{(r,\varphi) \mid 0 \le r \le R_1, \varphi = \varphi_2\}, \\ \partial A_{21} = \{(r,\varphi) \mid R_1 \le r \le R_2, \varphi = \varphi_1\}, \quad \partial A_{22} = \{(r,\varphi) \mid R_1 \le r \le R_2, \varphi = \varphi_2\}, \\ \partial A_3 = \{(r,\varphi) \mid r = R_2, \varphi_1 \le \varphi \le \varphi_2\} \end{gathered} \tag{1}$$

The common boundary curve of A_1 and A_2 is denoted by $\partial A_c = \{(r,\varphi) \mid r = R_1,$ $\varphi_1 \le \varphi \le \varphi_2\}$. The shear moduli of the cylindrically orthotropic material in radial and circumferential directions are as follows [1–3, 6]

$$G_r = G_{1r}, \quad G_\varphi = G_{1\varphi}, \text{ in } A_1, \quad G_r = G_{2r}, \quad G_\varphi = G_{2\varphi} \text{ in } A_2. \tag{2}$$

The connection between the cross sections A_1 and A_2 is perfect which means that the displacements and the radial shearing stress field are continuous in the whole cross section A. In the present problem the Prandtl's stress function formulation of the considered Saint-Venant torsion leads to the next boundary-value problem [1–5,9,10,12–14]

$$\frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r}\frac{\partial U_1}{\partial r} + \frac{g_1}{r^2}\frac{\partial^2 U_1}{\partial \varphi^2} = -2G_{1\varphi} \text{ in } A_1, \quad g_1 = \frac{G_{1\varphi}}{G_{1r}} \text{ in } A, \tag{3}$$

$$U_1 = 0 \text{ on } \partial A_{11} \cup \partial A_{12}, \quad \lim_{r \to 0} U_1(r,\varphi) = 0, \quad \varphi_1 \le \varphi \le \varphi_2, \tag{4}$$

$$\frac{\partial^2 U_2}{\partial r^2} + \frac{1}{r}\frac{\partial U_2}{\partial r} + \frac{g_2}{r^2}\frac{\partial^2 U_2}{\partial \varphi^2} = -2G_{2\varphi} \text{ in } A_2, \quad g_2 = \frac{G_{2\varphi}}{G_{2r}} \text{ in } A, \tag{5}$$

$$U_2 = 0 \text{ on } \partial A_{21} \cup \partial A_{22}, \quad U_2(R_2,\varphi) = 0, \quad \varphi_1 \le \varphi \le \varphi_2, \tag{6}$$

$$U_1(R_1,\varphi) = U_2(R_1,\varphi), \quad \varphi_1 \le \varphi \le \varphi_2, \tag{7}$$

$$\frac{1}{G_{1\varphi}} \frac{\partial U_1}{\partial r} = \frac{1}{G_{2\varphi}} \frac{\partial U_2}{\partial r}, \quad r = R_1, \quad \varphi_1 \le \varphi \le \varphi_2. \tag{8}$$

Eqs. (3) and (5) formulate the strain compatibility condition in terms of stress functions. The boundary conditions (4) and (6) express that the whole boundary contour ∂A is traction free. The continuity condition of radial shearing stresses on the curve ∂A_c is given by Eq. (7). Eq. (8) formulates the continuity conditions of the axial displacement field on the curve ∂A_c. The connections between the Prandtl's stress functions and torsion functions $\omega_i = \omega_i(r,\varphi)$ $(i = 1,2)$ are described by the next system of equations [1,2,4]

$$G_{1r} \frac{\partial \omega_1}{\partial r} = \frac{1}{r} \frac{\partial U_1}{\partial \varphi}, \quad G_{1\varphi}\left(\frac{1}{r} \frac{\partial \omega_1}{\partial \varphi} + r \right) = -\frac{\partial U_1}{\partial r}, \quad (r,\varphi) \in A_1, \tag{9}$$

$$G_{2r} \frac{\partial \omega_2}{\partial r} = \frac{1}{r} \frac{\partial U_2}{\partial \varphi}, \quad G_{2\varphi}\left(\frac{1}{r} \frac{\partial \omega_2}{\partial \varphi} + r \right) = -\frac{\partial U_2}{\partial r}, \quad (r,\varphi) \in A_2. \tag{10}$$

Eqs. (9) and (10) are based on formulae of shearing stresses which are as follows

$$\frac{\tau_{1rz}}{\vartheta} = G_{1r} \frac{\partial \omega_1}{\partial r} = \frac{1}{r} \frac{\partial U_1}{\partial \varphi}, \quad \frac{\tau_{1\varphi z}}{\vartheta} = G_{1\varphi}\left(\frac{1}{r} \frac{\partial \omega_1}{\partial r} + r \right) = -\frac{\partial U_1}{\partial r}, \tag{11}$$

$$\frac{\tau_{2rz}}{\vartheta} = G_{2r} \frac{\partial \omega_2}{\partial r} = \frac{1}{r} \frac{\partial U_2}{\partial \varphi}, \quad \frac{\tau_{2\varphi z}}{\vartheta} = G_{2\varphi}\left(\frac{1}{r} \frac{\partial \omega_2}{\partial r} + r \right) = -\frac{\partial U_2}{\partial r}. \tag{12}$$

In Eqs. (11) and (12) ϑ denotes the rate of twist with respect to axial coordinate z [1,2,5]. The connection between the applied torque T and ϑ is as follows

$$T = \vartheta S, \tag{13}$$

where S is the torsional rigidity of the cross section according to the theory of Saint-Venant torsion. We have [1–5]

$$S = 2\left(\int_{A_1} U_1 \mathrm{d}A + \int_{A_2} U_2 \mathrm{d}A \right). \tag{14}$$

3 SOLUTION OF THE TORSION PROBLEM

Solution of boundary value problem formulated by Eqs. (3–8) can be obtained by using the technique known as separation of variables. In our case, the application of the method of separation gives the next results for $U_1 = U_1(r,\varphi)$ and $U_2 = U_2(r,\varphi)$

$$U_1(r,\varphi)=\sum_{k=1}^{\infty}\left(a_{1k}r^{p_{1k}}+\frac{2G_{1\varphi}\alpha_k}{g_1\lambda_k^2-4}r^2\right)\sin\lambda_k\varphi,\ \ (r,\varphi)\in A_1, \tag{15}$$

$$U_2(r,\varphi)=\sum_{k=1}^{\infty}\left(a_{2k}r^{p_{2k}}+a_{3k}r^{-p_{2k}}+\frac{2G_{2\varphi}\alpha_k}{g_2\lambda_k^2-4}r^2\right)\sin\lambda_k\varphi,\ \ (r,\varphi)\in A_2. \tag{16}$$

Here,

$$p_{1k}=\lambda_k\sqrt{g_1},\ \ p_{2k}=\lambda_k\sqrt{g_2},\ \ \alpha_k=\frac{8}{(2k-1)\pi}, \tag{17}$$

a_{1k}, a_{2k}, a_{3k} are constants whose values are obtained from continuity condition (7) and the continuity conditions of radial shearing stresses and torsion functions. The system of linear equations for the unknown constants of Prandtl’s stress functions are as follows

$$a_{1k}R_1^{p_{1k}}-a_{2k}R_1^{p_{2k}}-a_{3k}R_1^{-p_{2k}}=\left(\frac{2G_{2\varphi}\alpha_k}{g_2\lambda_k^2-4}-\frac{2G_{1\varphi}\alpha_k}{g_1\lambda_k^2-4}\right)R_1^2, \tag{18}$$

$$a_{1k}\frac{p_{1k}}{G_{1\varphi}}R_1^{p_{1k}-1}-a_{2k}\frac{p_{2k}}{G_{2\varphi}}R_2^{p_{2k}-1}R_1^{p_{2k}-1}+a_{3k}\frac{p_{2k}}{G_{2\varphi}}R_2^{-p_{2k}-1}=\left(\frac{4\alpha_k}{g_2\lambda_k^2-4}-\frac{4\alpha_k}{g_1\lambda_k^2-4}\right)R_1, \tag{19}$$

$$a_{1k}R_2^{p_{2k}}+a_{3k}R_2^{-p_{2k}}=-\frac{2G_{2\varphi}\alpha_k}{g_2\lambda_k^2-4}R_2^2,\ \ (k=1,2,\ldots). \tag{20}$$

Determination of the torsion function is based on the solution of the next system of differential equations

$$\frac{\partial\omega_1}{\partial r}=\frac{1}{rG_{1r}}\frac{\partial U_1}{\partial\varphi},\ \ \frac{\partial\omega_1}{\partial\varphi}=-\frac{r}{G_{1\varphi}}\frac{\partial U_1}{\partial r}-r^2,\ \ (r,\varphi)\in A_1, \tag{21}$$

$$\frac{\partial\omega_2}{\partial r}=\frac{1}{rG_{2r}}\frac{\partial U_2}{\partial\varphi},\ \ \frac{\partial\omega_2}{\partial\varphi}=-\frac{r}{G_{2\varphi}}\frac{\partial U_2}{\partial r}-r^2,\ \ (r,\varphi)\in A_2. \tag{22}$$

The value of torsion function at point O (Fig. 1) can be prescribed an arbitrary value. We use the next choice $\omega(0,0)=0$ and we look for such solution of system of equations which satisfy the following conditions

$$\omega_1(R_1,\varphi)=\omega_2(R_1,\varphi),\ \ \varphi_1\le\varphi\le\varphi_2. \tag{23}$$

From Eqs. (21) and (22) it follows that the torsion function $\omega_1=\omega_1(r,\varphi)$ and $\omega_2=\omega_2(r,\varphi)$ satisfy the stress boundary conditions

$$G_{1\varphi}\left(\frac{1}{r}\frac{\partial\omega_1}{\partial\varphi}+r\right)=0\ \text{on}\ \partial A_{11}\cup\partial A_{12}, \tag{24}$$

$$G_{2\varphi}\left(\frac{1}{r}\frac{\partial \omega_2}{\partial \varphi}+r\right)=0 \text{ on } \partial A_{21}\cup\partial A_{22}, \tag{25}$$

$$G_{2r}\frac{\partial \omega}{\partial r}=0 \text{ on } \partial A_3, \tag{26}$$

and they fulfil the continuity conditions of radial stress field which is

$$G_{1r}\frac{\partial \omega_1}{\partial r}=G_{2r}\frac{\partial \omega_2}{\partial r} \text{ on } \partial A_c. \tag{27}$$

The torsional rigidity of the composite cross section can be computed by the application of next formula [1–4]

$$S=2\left(\int_{A_1} U_1(r,\varphi)\,\mathrm{d}A+\int_{A_2} U_2(r,\varphi)\,\mathrm{d}A\right). \tag{28}$$

4 NUMERICAL EXAMPLES

The following data are used in the presented numerical example: $R_1=0.035\ [\mathrm{m}]$, $R_2=0.06\ [\mathrm{m}]$, $\varphi_1=0$, $\varphi_2=\frac{\pi}{3}$, $G_{1r}=2.5\times10^8\ [\mathrm{Pa}]$, $G_{1\varphi}=5\times10^8\ [\mathrm{Pa}]$, $G_{2r}=4\times10^9\ [\mathrm{Pa}]$, $G_{2\varphi}=7.5\times10^9\ [\mathrm{Pa}]$, $\vartheta=10^{-2}\left[\frac{1}{\mathrm{m}}\right]$. Fig. 2 shows the plots of Prandtl's stress function for different values of φ as a function of radial coordinate. In Fig. 3 the graphs of shearing stress $\tau_{rz}(r,\varphi)$ are shown for different values of polar angle as a function of radial coordinate. Fig. 4 illustrates the graphs of the shearing stresses $\tau_{\varphi z}(r,\varphi)$ as a function of r for different values of polar angle. Fig. 5 shows the graphs of $\tau_{\varphi z}(r,\varphi)$ as a function of φ for three different values of radial coordinate. Application of formula (14) gives the next result for the torsional rigidity of composite cross section

$$S=2476.36978\ \left[\mathrm{Nm}^2\right]. \tag{29}_{1,2}$$

The level lines of the Prandlt's stress function is shown in Fig. 6.

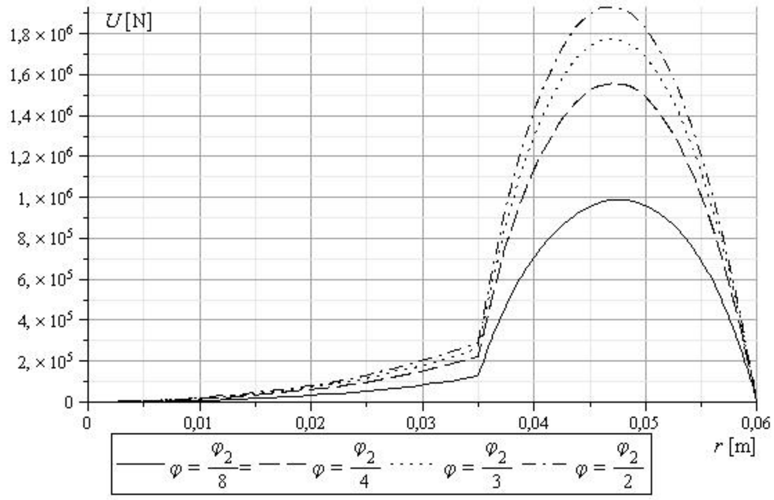

Fig. 2
The plots of the Prandtl's stress function for different values of φ as a function of r

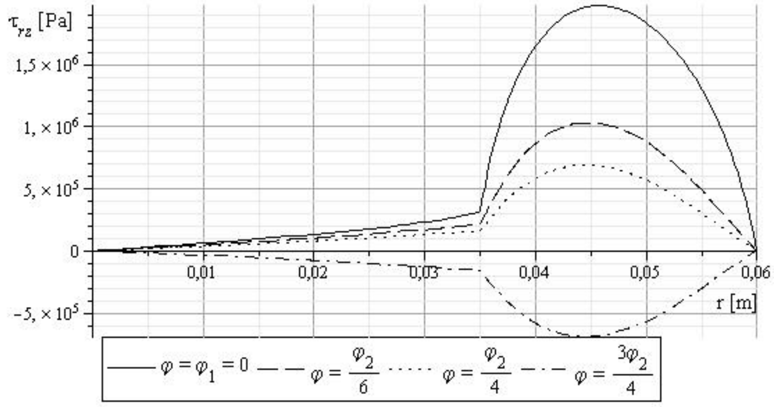

Fig. 3
The graphs of shearing stresses $\tau_{rz}(r,\varphi)$ for different values of φ as a function of r

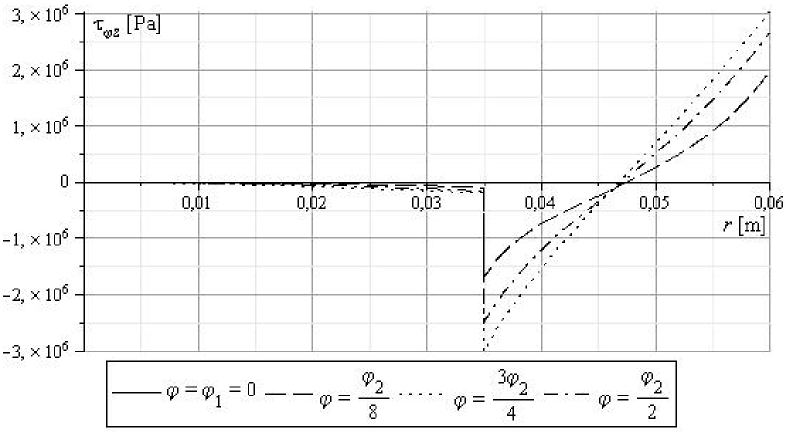

Fig. 4
The graphs of shearing stresses $\tau_{\varphi z}(r,\varphi)$ for different values of φ as a function of r

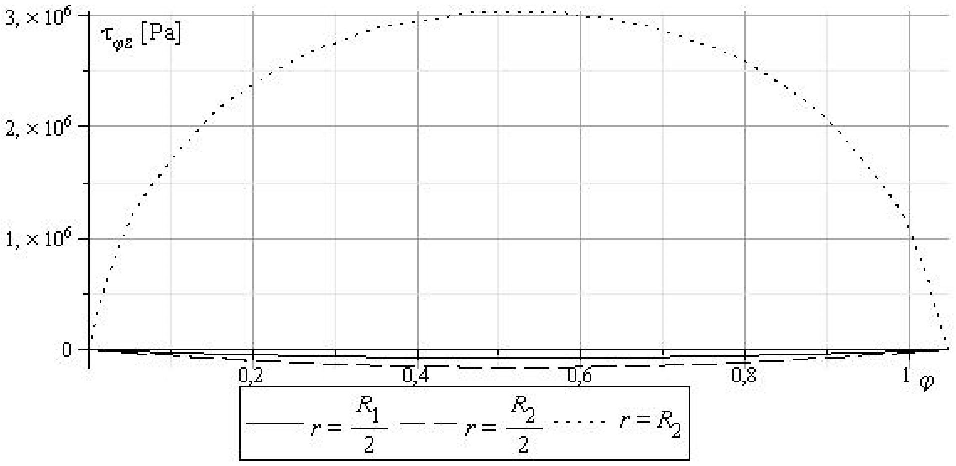

Fig. 5
The graphs of shearing stresses $\tau_{\varphi z}(r,\varphi)$ for different values of r as a function of φ

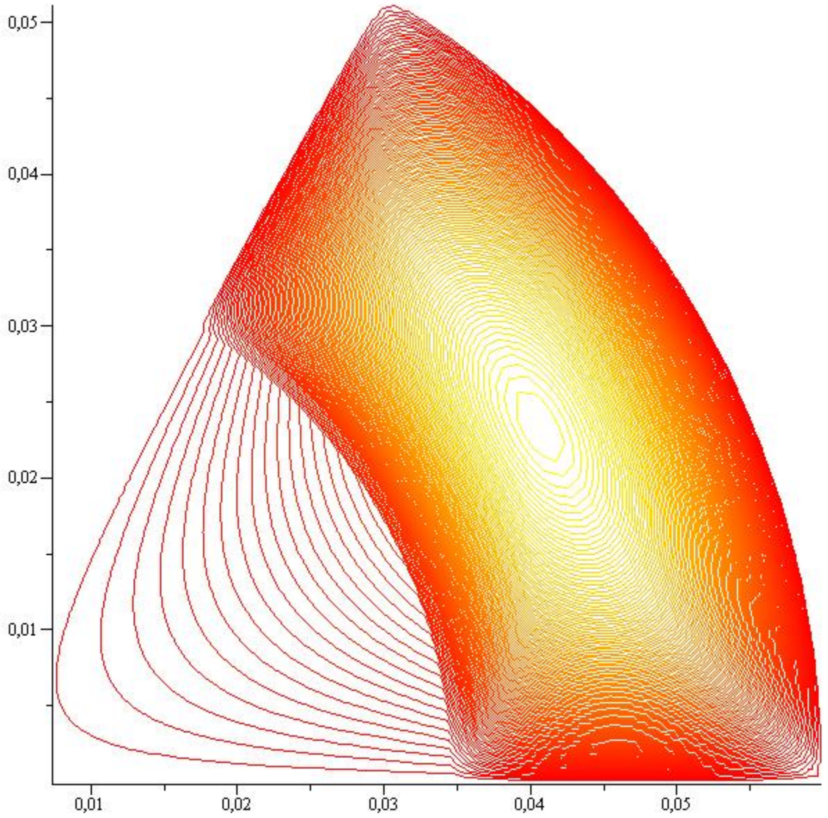

Fig. 6
The level lines of the Prandtl's stress function

5 CONCLUSIONS

An analytical solution is developed to solve the Saint-Venant's torsion of cylindrically orthotropic bar with a cross section of sector of solid circle. The cross section of the considered bar consists of two different cylindrically anisotropic material. The cross-sectional inhomogeneity depends only on the radial coordinate. The numerical results of the paper can be used as benchmark solution to check the accuracy of the different approximate methods such as finite element method, or boundary element method, etc.

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