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DETERMINATION OF THE PRANDTL'S STRESS FUNCTION FOR NON-WARPING ANISOTROPIC CROSS SECTION

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Abstract

The object of this paper is the Saint-Venant torsion of homogeneous anisotropic cross section. The classes of anisotropy considered has at least one plane of elastic symmetry, which is normal to the axis of the beam. A new and very simple derivation is given to obtain the boundary contour of the non-warping anisotropic cross section. The determination of the torsional rigidity in terms of area of the cross section is also presented.

1 INTRODUCTION

The object of this paper is the Saint-Venant (uniform) torsion of the homogeneous anisotropic cross section. The Saint-Venant torsion of anisotropic linear elastic beams has been the subject of several studies from both theoretical and numerical viewpoints. Books by Sokolnikoff [1], Lekhnitskii [2,3], Sadd [4], Rand and Rovenski [5] give the detailed analyses for problem of the torsion of anisotropic elastic beams. In these books both the torsion function formulation and Prandtl's stress function formulation are presented. The non-warping property of anisotropic cross section is discussed in papers by Chen [6], Ecsedi [7] and Horgan [8]. In this article a very simple derivation is given to obtain the equation of the boundary contour of the twisted non-warping anisotropic cross section. The classes of anisotropy considered have at least one plane of elastic symmetry, which is normal to the axes z (monoclinic material) (Fig. 1). In this case of anisotropy there is no coupling between the bending and torsional deformations [9]. Chen and Wei [10] prove the next theorem: Of all simply connected cross section with a given cross-sectional area and the same shear rigidities an elliptical cross section with zero warping has the maximum torsional rigidity. This paper gives the expression of torsional rigidity in terms of cross-sectional area and shear rigidity for non-warping anisotropic elliptical cross section.

2 DETERMINATION OF THE PRANDTL'S STRESS FUNCTION

We consider the Saint-Venant torsion of the anisotropic homogeneous linear elastic beam with solid cross section (Fig. 1). The Cartesian coordinate system $Oxyz$ is positioned at the left end cross section of the beam as shown in Fig. 1. The torsion

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function and twist are denoted by $\omega = \omega(x, y)$ and ϑ, respectively, the length of the beam is L furthermore the applied torque is T (Fig. 1). It is known that [1–4]

$$\tau_{xz} = \vartheta \frac{\partial U}{\partial y}, \quad \tau_{yz} = -\vartheta \frac{\partial U}{\partial x}, \tag{1}$$

$$\gamma_{xz} = \vartheta\left(\frac{\partial \omega}{\partial x} - y\right), \quad \gamma_{yz} = \vartheta\left(\frac{\partial \omega}{\partial y} + x\right). \tag{2}$$

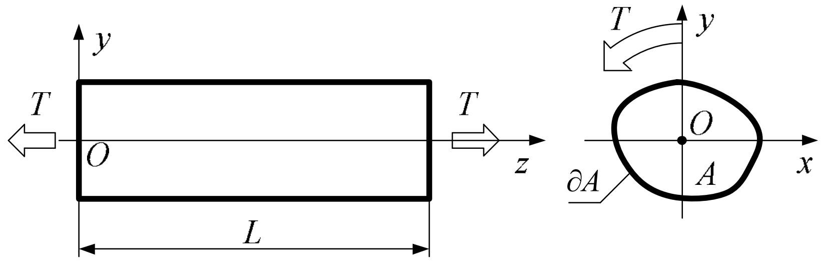

Fig. 1
Saint-Venant torsion of homogeneous anisotrpic bar.

In Eq. (1) $U = U(x, y)$ is the Prandtl's stress function which vanishes on the boundary curve ∂A of the cross section. τ_{xz}, τ_{yz} are the shearing stresses, γ_{xz} and γ_{yz} are the shear strains. In the present case the constitutive equations for torsional deformation can be formulated as [1–5]

$$\tau_{xz} = A_{55}\gamma_{xz} + A_{54}\gamma_{yz}, \quad \tau_{yz} = A_{45}\gamma_{xz} + A_{44}\gamma_{yz}, \tag{3}$$

where A_{55}, A_{44}, $A_{45} = A_{54}$ are the shear rigidity of the monoclinic elastic material and according to the non-negativity of the specific strain energy we have [2,3,5]

$$A_{44} > 0, \quad A_{55} > 0, \quad A_{44}A_{55} - A_{45}^2 > 0. \tag{4}$$

For non-warping cross section $\omega(x, y) = 0$, that is

$$\frac{\partial U}{\partial x} = A_{45}y - A_{44}x, \quad \frac{\partial U}{\partial y} = -A_{55}y + A_{45}x. \tag{5}$$

Integration of Eq. $(5)_1$ gives

$$U(x, y) = A_{45}xy - A_{44}\frac{x^2}{2} + f(y). \tag{6}$$

Here $f = f(y)$ an arbitrary function of y since the integration was carried out with respect to x. Combination of Eq. $(5)_2$ with (6) yields the next result

$$A_{45}x + \frac{\mathrm{d}f}{\mathrm{d}y} = -A_{55}y + A_{45}x. \tag{7}$$

From Eq. (7) it follows that

$$f(y) = -A_{55}\frac{y^2}{2} + \frac{C^2}{2}. \tag{8}$$

In Eq. (8) C is the constant of integration. Substitution of Eq. (8) into Eq. (6) gives the expression of the Prandtl's stress function

$$U(x,y) = \frac{C^2}{2}\left(1 - \frac{A_{44}}{C^2} + 2\frac{A_{45}}{C^2}xy - \frac{A_{55}}{C^2}y^2\right). \tag{9}$$

The boundary contour of the cross section is prescribed by the next equation

$$U(x,y) = 0, \tag{10}$$

that is

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = 1 \text{ on } \partial A, \tag{11}$$

where

$$a_{11} = \frac{A_{44}}{C^2}, \quad a_{12} = a_{21} = -\frac{A_{45}}{C^2}, \quad a_{22} = \frac{A_{55}}{C^2}. \tag{12}$$

It is obvious according to (4)

$$a_{11} > 0, \quad a_{22} > 0, \quad a_{11}a_{22} - a_{12}^2 > 0. \tag{13}$$

From inequality relations (13) it follows that the boundary contour of the non-warping anisotropic cross section is an ellipse. To obtain the axis of ellipse whose equation is given by (11) we introduce new coordinates by the next definition (Fig. 2)

$$X = x\cos\alpha + y\sin\alpha, \quad Y = -x\sin\alpha + y\cos\alpha. \tag{14}$$

The canonical implicit equation of ellipse is obtained for the next value of α [11]

$$\tan\alpha = \frac{\lambda_1 - a_{11}}{a_{12}}, \quad \tan\left(\alpha + \frac{\pi}{2}\right) = \frac{\lambda_2 - a_{11}}{a_{12}}, \tag{15}$$

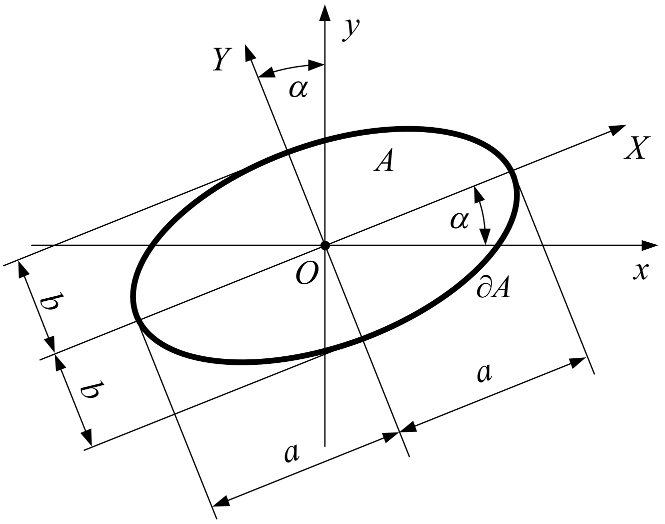

Fig. 2
Elliptical cross section

where

$$\lambda_1 = \frac{1}{2}\left(a_{11} + a_{22} + \sqrt{\left(a_{11} - a_{22}\right)^2 - 4a_{12}^2}\right), \tag{16}$$

$$\lambda_2 = \frac{1}{2}\left(a_{11} + a_{22} - \sqrt{\left(a_{11} - a_{22}\right)^2 - 4a_{12}^2}\right) \tag{17}$$

in terms of X and Y

$$\lambda_1 X^2 + \lambda_2 Y^2 = 1. \tag{18}$$

It is evident that the area of the elliptical cross section shown in Fig. 2 is

$$A = ab\pi = \frac{\pi}{\sqrt{\lambda_1 \lambda_2}}, \quad a = \frac{1}{\sqrt{\lambda_1}}, \quad b = \frac{1}{\sqrt{\lambda_2}}. \tag{19}$$

Combination of Eq. (11) with Eq. (16) and (17) and substituting the obtained results for λ_1 and λ_2 into Eq. (19) gives the following formula for A

$$A = \frac{C^2\pi}{\sqrt{A_{44}A_{55} - A_{45}^2}}. \tag{20}$$

3 TORSIONAL RIGIDITY

At first we give the expression of the Prandtl's stress function for the solid elliptical cross section shown in Fig. 2

$$U(x,y)=\frac{C^2}{2}\left(1-\lambda_1X^2-\lambda_2Y^2\right). \tag{21}$$

The torsional rigidity S is defined by the next equation

$$S=\frac{T}{\vartheta}. \tag{22}$$

Accordint to theory of Saint-Venant torsion we have [1–5]

$$S=2\int_A U(x,y)\mathrm{d}x\mathrm{d}y=2\int_A U(X,Y)J\mathrm{d}X\mathrm{d}Y, \tag{23}$$

where

$$J=\frac{\partial(x,y)}{\partial(X,Y)}=1 \tag{24}$$

denotes the determinant of Jacobian matrix of transformation given by Eq. (14). Substitution of Eq. (21) into formula (23) leads to the next result

$$S=\frac{A^2}{2\pi}\sqrt{A_{44}A_{55}-A_{45}^2}. \tag{25}$$

In the derivation of formula (25) the next equations have been used

$$\int_A 1\mathrm{d}X\mathrm{d}Y=A, \tag{26}$$

$$\int_A \lambda_1X^2\mathrm{d}X\mathrm{d}Y=\lambda_1\int_A X^2\mathrm{d}X\mathrm{d}Y=\lambda_1\frac{\pi}{4\sqrt{\lambda_1^3\lambda_2}}=\frac{\pi}{4\sqrt{\lambda_1\lambda_2}}=\frac{A}{4}, \tag{27}$$

$$\int_A \lambda_2Y^2\mathrm{d}X\mathrm{d}Y=\lambda_2\int_A Y^2\mathrm{d}X\mathrm{d}Y=\lambda_2\frac{\pi}{4\sqrt{\lambda_1\lambda_2^3}}=\frac{\pi}{4\sqrt{\lambda_1\lambda_2}}=\frac{A}{4}. \tag{28}$$

From the theorem of Chen and Wei [10], which says that of all simply connected cross section with a given cross sectional area A and shear rigidities A_{44}, A_{55}, $A_{45}=A_{54}$ the elliptical cross section with zero warping has the maximum torsional rigidity, the next upper bound follows for the torsional rigidty

$$S\le S_U=\frac{A^2}{2\pi}\sqrt{A_{44}A_{55}-A_{45}^2}. \tag{29}$$

Equality in relation (29) is valid only if the equation of the boundary contour is

$$A_{44}x^2 - 2A_{45}xy + A_{55}y^2 = \frac{A}{\pi}\sqrt{A_{44}A_{55} - A_{45}^2}. \tag{30}$$

4 CIRCULAR CROSS SECTION

It is known that the homogeneous isotropic circular cross section is not warped [1,4,9]. Let R be the radius of the boundary circle and denote the shear modulus G of the isotropic circular cross section. In this case we have

$$\begin{gathered} A_{44} = A_{55} = G, \quad A_{45} = A_{54} = 0, \quad A = R^2\pi, \quad \lambda_1 = \lambda_2 = \frac{\pi}{A} = \frac{1}{R^2}, \\ C^2 = \frac{GA}{\pi} = GR^2, \quad x = X, \quad y = Y. \end{gathered} \tag{31}$$

For the Prandtl's stress function from Eqs (21) and (31) we obtain

$$U(x,y) = \frac{GR^2}{2}\left(1 - \frac{x^2}{R^2} - \frac{y^2}{R^2}\right) = \frac{G}{2}\left(R^2 - x^2 - y^2\right). \tag{32}$$

The torsional rigidity of the isotropic cross section can be computed by the application of Eq. (25)

$$S = G\frac{R^4\pi}{2}. \tag{33}$$

Results obtained for the Prandtl's stress function and torsional rigidity are the same as we can find in the textbooks of elasticity [1,4,9].

5 EXAMPLES FOR THE APPLICATION OF UPPER BOUND FORMULA

In this section two examples illustrate the application of upper bound formula (29) to orthotropic elliptic and square cross sections. For orthotropic elastic material we have $A_{45} = A_{54} = 0$.

5.1 Elliptical cross section

The principal directions of orthotropy are the axes x and y as shown in Fig. 3. Exact value of torsional rigidity of orthotropic elliptical cross section is [2,3,5]

$$S = \pi a^3 b^3 \frac{A_{44}A_{55}}{A_{44}a^2 + A_{55}b^2}. \tag{34}$$

In the present case the upper bound formula (29) gives

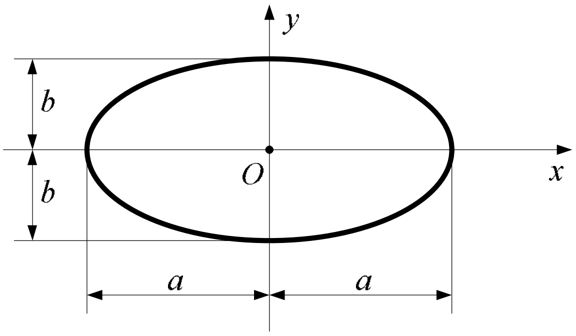

Fig. 3
Orthotropic elliptical cross section

$$S \le S_U = \frac{a^2 b^2 \pi}{2} \sqrt{A_{44} A_{55}}. \tag{35}$$

Let

$$a^2 = \mu b^2 \text{ and } A_{55} = \lambda A_{44} \tag{36}$$

be and we define the functions $f_1 = f_1(\lambda)$ and $F_1 = F_1(\lambda)$ as

$$f_1(\lambda) = \frac{S}{\mu b^4 A_{44} \pi} = \frac{\lambda \sqrt{\mu}}{\mu + \lambda}, \tag{37}$$

$$F_1(\lambda) = \frac{S_U}{\mu b^4 A_{44} \pi} = \frac{\sqrt{\lambda}}{2}. \tag{38}$$

The graphs of $f_1 = f_1(\lambda)$ and $F_1 = F_1(\lambda)$ are shown in Fig. 4 for $\mu = 2$. Here we note if $F_1(\lambda) = f_1(\lambda)$ then we have

$$\frac{1+\mu}{2} = \sqrt{\lambda \mu}, \tag{39}$$

that is $\mu = \lambda$ according to the result of Chen [6].

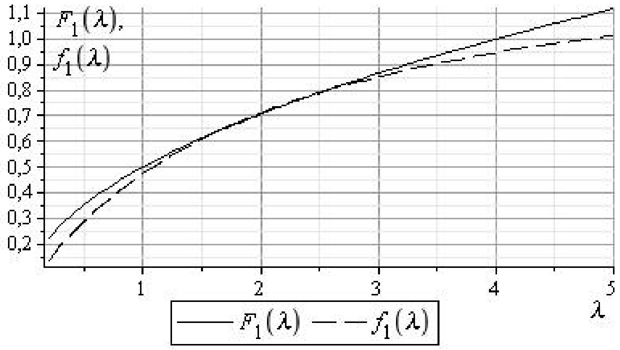

Fig 4.
The graphs of $f_1 = f_1(\lambda)$ and $F_1 = F_1(\lambda)$

5.2 Square cross section

The Prandtl's stress function of orthotropic square cross section (Fig. 5) can be represented as [2,3]

$$U(x,y)=\frac{32a^2}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)(2n-1)}\frac{\sin\frac{(2m-1)\pi}{a}x\sin\frac{(2n-1)\pi}{a}y}{\frac{(2m-1)^2}{A_{55}}+\frac{(2n-1)^2}{A_{44}}}. \tag{40}$$

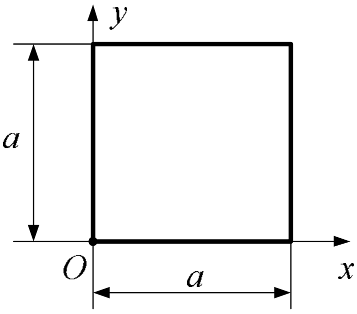

Fig. 5
Orthotropic square cross section

Exact value of the torsional rigidity of orthotropic square cross section is obtained from [1–3,5] the formula

$$S=2\int_A U(x,y)\,\mathrm{d}A. \tag{41}$$

A detailed computation gives the next result

$$S=\frac{256a^4}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\frac{(2m-1)^2}{A_{55}}+\frac{(2n-1)^2}{A_{44}}}. \tag{42}$$

The upper bound formula (29) gives

$$S\le S_U=\frac{a^4}{2\pi}\sqrt{A_{44}A_{55}}. \tag{43}$$

Let λ be defined as

$$\lambda=\frac{A_{55}}{A_{44}} \tag{44}$$

and we introduce the functions $F_2=F_2(\lambda)$ and $f_2=f_2(\lambda)$

$$F_2(\lambda)=\frac{S_U}{A_{44}a^4}=\frac{\sqrt{\lambda}}{2\pi}, \tag{45}$$

$$f_2(\lambda)=\frac{256}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left[(2m-1)^2+(2n-1)^2\right]\left[\frac{(2m-1)^2}{\lambda}+(2n-1)^2\right]}. \tag{46}$$

The graphs of $F_2=F_2(\lambda)$ and $f_2=f_2(\lambda)$ are illustrated in Fig. 6. The best lower bound is obtained for $\lambda_0=0.7298156$ (Fig. 7). This result can be derived from the equation

$$\frac{\mathrm{d}F_2}{\mathrm{d}\lambda}-\frac{\mathrm{d}f_2}{\mathrm{d}\lambda}=0. \tag{47}$$

By a simple calculation we get

$$F_2(\lambda_0)=0.1359648,\quad f_2(\lambda_0)=0.1187218. \tag{48}$$

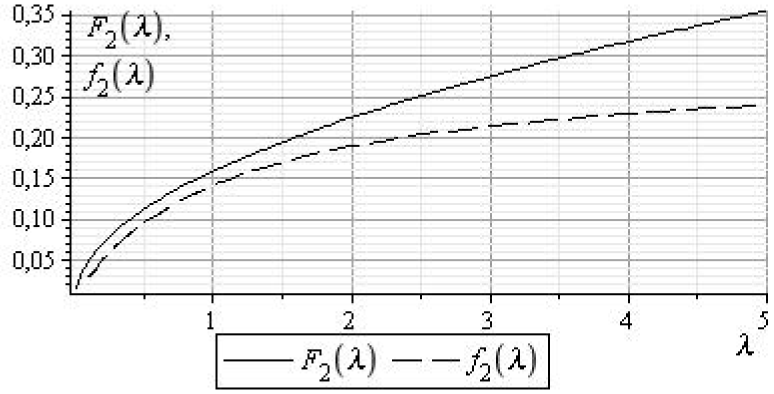

Fig. 6
The graphs of $F_2=F_2(\lambda)$ and $f_2=f_2(\lambda)$

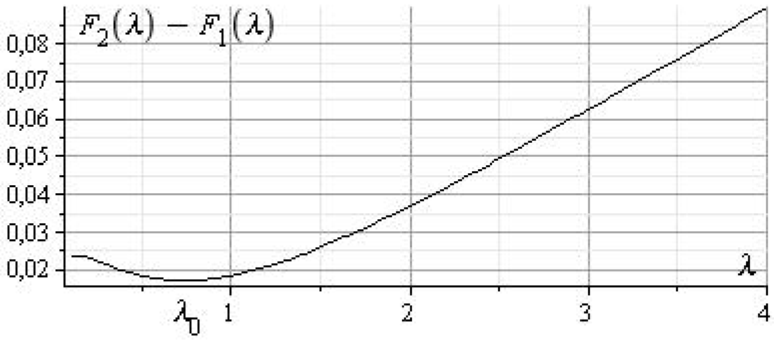

Fig 7.
Determination of λ_0

6 CONCLUSIONS

This paper deals with the Saint-Venant torsion of anisotropic solid cross section. A primary method is used to determine the shape of the non-warping anisotropic cross section. The torsional rigidity of the non-warping cross section is given in terms of cross-sectional area and shear rigidities.

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