



GREEN FUNCTIONS FOR SOME THREE POINT BOUNDARY VALUE PROBLEMS WITH APPLICATIONS

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Abstract: *The present paper is devoted to the issue of the Green functions that are related to some three point boundary and eigenvalue problems. A detailed definition is given for the Green functions provided that the considered three point boundary value problems are governed by a class of ordinary differential equations which are associated with homogeneous boundary and continuity conditions. The definition is a constructive one, i.e., it provides the means that are needed for calculating the Green functions. The fundamental properties of the Green functions – existence, symmetry properties, etc. – are also clarified. Making use of these Green functions, a class of the three point eigenvalue problems can be reduced to eigenvalue problems governed by homogeneous Fredholm integral equations. A solution algorithm is also presented.*

Keywords: *Three point boundary value problems, Green functions, eigenvalue problems, Fredholm integral equations*

1. INTRODUCTION

The first paper in which the concept of the Green functions appeared was published in 1901 [1]. This concept was generalized in [2] while the first book that systematically contained the concept of the Green function was published in 1926 [3]. Books [4, 5] by Collatz contain the definition of the Green function for ordinary linear differential equations, clarifies their most important properties and a number of Green functions are presented in closed forms. The concept of the Green function was generalized for a class of ordinary differential equation systems by introducing the Green function matrices in [6]. As regards the degenerated ordinary differential equation systems, the results of Obádovics have been developed further in thesis [7]. Some applications of the Green function matrices have recently been published in [8]. Further textbooks devoted to the issue of the Green functions have also been published since the books by Collatz appeared: [9, 10, 11]. Existence proof for some three point boundary value problems related to third order nonlinear differential equations is given in paper [12] by using Green functions. For some three point boundary value problems governed by linear ordinary differential equations of order two the corresponding Green functions are presented in paper [13].

This paper is organized in eight sections. Section 2 presents the differential operator, the boundary and continuity conditions, clarifies the concept of self adjointness and introduces the Rayleigh quotient for the three point eigenvalue problems. Section 3 provides a definition for the Green function of three point boundary value problems. The definition is a constructive one since it makes it possible to reply to the issue of how to calculate the Green functions – the procedure is detailed in Section 4. The most important properties of the Green functions are presented in Section 5 where an existence theorem is also proved. The three examples that are related to the boundary value problems of heterogeneous beams demonstrate the calculation steps of the Green functions in Section 6. Self-adjoint three point eigenvalue problems can be reduced to eigenvalue problems governed by homogeneous Fredholm integral equations with the symmetric Green function as the kernel. Section 7 presents a solution algorithm for finding approximate solutions for the eigenvalue problems related to Fredholm integral equations. The last section contains the conclusions.

2. A CLASS OF THE THREE POINT BOUNDARY VALUE PROBLEMS

2.1. The differential operator with the boundary and continuity conditions. Consider the ordinary differential equation¹

$$K[y] = \lambda M[y] \quad (1a)$$

where $y(x)$ is the unknown function, λ is a parameter (the eigenvalue sought). The differential operators $K[y]$ and $M[y]$ are defined by the following relationships:

$$\begin{aligned} K[y] &= \sum_{n=0}^{\kappa} (-1)^n \left[f_n(x) y^{(n)}(x) \right]^{(n)}, & d^n(\dots)/dx^n &= (\dots)^{(n)}; \\ M[y] &= \sum_{n=0}^{\mu} (-1)^n \left[g_n(x) y^{(n)}(x) \right]^{(n)}, & \kappa > \mu &\geq 1 \end{aligned} \quad (1b)$$

in which the real function $(f_\nu(x)) [g_\nu(x)]$ is differentiable continuously $(\kappa) [\mu]$ times and

$$f_\kappa(x) \neq 0, \quad g_\mu(x) \neq 0 \quad \text{if } x \in [a, c]. \quad (1c)$$

Note that 2κ , which is the order of the differential operator on the left side of (1a), is greater than 2μ , which is the order of the differential operator on the right side. Let $x \in [a, c]$, $c > a$, $c - a = \ell$ be the interval in which we seek the solution of differential equation (1a). Further let b be an inner point in the interval $[a, c]$: $b \in [a, b]$, $b - a = \ell_1$, $c - b = \ell_2$, $\ell_1 + \ell_2 = \ell$.

Some quantities in the intervals $[a, b]$ and $[b, c]$ are denoted by the Latin I and II . Accordingly y_I and y_{II} are the solutions to the differential equation (1) in the intervals I and II .

We shall assume that differential equation (1) is associated with the following boundary and continuity conditions:

$$U_{ar}[y] = \sum_{n=1}^{2\kappa} \alpha_{nrI} y_I^{(n-1)}(a) = 0, \quad r = 1, 2, \dots, \kappa \quad (2a)$$

$$\begin{aligned} U_{br}[y] &= U_{brI}[y_I] - U_{brII}[y_{II}] = \\ &= \sum_{n=1}^{2\kappa} \left(\beta_{nrI} y_I^{(n-1)}(b) - \beta_{nrII} y_{II}^{(n-1)}(b) \right) = 0, \quad r = 1, 2, \dots, 2\kappa \end{aligned} \quad (2b)$$

$$U_{cr}[y] = \sum_{n=1}^{2\kappa} \gamma_{nrII} y_{II}^{(n-1)}(a) = 0, \quad r = 1, 2, \dots, \kappa \quad (2c)$$

where α_{nrI} , β_{nrI} , β_{nrII} and γ_{nrII} are nonzero real constants.

Differential equation (1) with the boundary and continuity conditions (2) constitute an eigenvalue problem in which λ is the eigenvalue to be determined. If $\mu = 0$ the right side of differential equation (1) is of the form

$$M[y] = g_0(x)y(x) \quad (3)$$

and the eigenvalue problem is called simple. In what follows it is assumed that $g_0(x) > 0$ if $x \in [a, c]$.

The eigenvalue problems that provide the eigenfrequencies for the longitudinal and torsional vibrations of rods as well as for the transverse vibrations of strings and beams are all simple ones. If $\mu > 0$ the eigenvalue problem is called generalized eigenvalue problem.

The boundary and continuity conditions should be linearly independent of each other.

It is obvious that a linear combination of the boundary conditions is also boundary condition. By selecting suitable linear combinations we may remove derivatives with order higher than $\kappa - 1$ from some boundary conditions. If we do that in all possible way we may have altogether, say, e boundary conditions which do not involve derivatives higher than $\kappa - 1$. These boundary conditions are called essential boundary conditions. The further $2\kappa - e$ boundary conditions are the natural boundary conditions [14].

The functions $u(x)$ and $v(x)$ ($u(x), v(x)$ are not identically equal to zero if $x \in [a, b]$) are called comparison functions if they satisfy the boundary and continuity conditions and are eigenfunctions if they, in addition to this, satisfy differential equation (1) as well.

2.2. Self-adjointness. The integrals

$$(u, v)_K = \int_a^c u(x) K[v(x)] dx, \quad (u, v)_M = \int_a^c u(x) M[v(x)] dx \quad (4)$$

¹The line of thought of the present section is based in many respects on book [4] by Lothar Collatz (1910-1990).

taken on the set of the comparison functions $u(x)$, $v(x)$ are products defined on the operators K and M . Let us now detail the product $(u, v)_K$. Making use of (1b) we may write:

$$(u, v)_K = \int_a^b u(x) \sum_{n=0}^{\kappa} (-1)^n [f_n(x)v^{(n)}(x)]^{(n)} dx + \int_b^c u(x) \sum_{n=0}^{\kappa} (-1)^n [f_n(x)v^{(n)}(x)]^{(n)} dx \quad (5)$$

in which the integral

$$I_{ab} = (-1)^n \int_a^b u(x) [f_n(x)v^{(n)}(x)]^{(n)} dx$$

can be manipulated into a more suitable form if we perform partial integrations. We get

$$\begin{aligned} I_{ab} &= (-1)^n \left[u(x) [f_n(x)v^{(n)}(x)]^{(n-1)} \right]_a^{b-0} + (-1)^{n-1} \int_a^b u(x) [f_n(x)v^{(n)}(x)]^{(n-1)} dx = \\ &= (-1)^n \left[u(x) [f_n(x)v^{(n)}(x)]^{(n-1)} - u(x)^{(1)} [f_n(x)v^{(n)}(x)]^{(n-2)} + \right. \\ &\quad \left. + u(x)^{(2)} [f_n(x)v^{(n)}(x)]^{(n-3)} - \dots \right]_a^{b-0} + \int_a^b u^{(n)}(x) f_n(x) v^{(n)}(x) dx = \\ &= \left[\sum_{r=0}^{n-1} (-1)^{(n+r)} u(x)^{(r)} [f_n(x)v^{(n)}(x)]^{(n-1-r)} \right]_a^{b-0} + \int_a^b u^{(n)}(x) f_n(x) v^{(n)}(x) dx. \end{aligned} \quad (6)$$

Hence

$$\begin{aligned} (u, v)_K &= \left[\sum_{n=0}^{\kappa} \sum_{r=0}^{n-1} (-1)^{(n+r)} u(x)^{(r)} [f_n(x)v^{(n)}(x)]^{(n-1-r)} \right]_a^{b-0} + \\ &\quad + \left[\sum_{n=0}^{\kappa} \sum_{r=0}^{n-1} (-1)^{(n+r)} u(x)^{(r)} [f_n(x)v^{(n)}(x)]^{(n-1-r)} \right]_{b+0}^c + \\ &\quad + \sum_{n=0}^{\kappa} \int_a^c u^{(n)}(x) f_n(x) v^{(n)}(x) dx = \\ &= K_0(u, v) + \sum_{n=0}^{\kappa} \int_a^c u^{(n)}(x) f_n(x) v^{(n)}(x) dx. \end{aligned} \quad (7a)$$

It follows from equation (7a) that

$$\begin{aligned} (u, v)_M &= \left[\sum_{n=0}^{\mu} \sum_{r=0}^{n-1} (-1)^{(n+r)} u(x)^{(r)} [g_n(x)v^{(n)}(x)]^{(n-1-r)} \right]_a^{b-0} + \\ &\quad + \left[\sum_{n=0}^{\mu} \sum_{r=0}^{n-1} (-1)^{(n+r)} u(x)^{(r)} [g_n(x)v^{(n)}(x)]^{(n-1-r)} \right]_{b+0}^c + \\ &\quad + \sum_{n=0}^{\mu} \int_a^c u^{(n)}(x) g_n(x) v^{(n)}(x) dx = \\ &= M_0(u, v) + \sum_{n=0}^{\mu} \int_a^c u^{(n)}(x) g_n(x) v^{(n)}(x) dx. \end{aligned} \quad (7b)$$

These results are naturally valid for the products $(v, u)_K$ and $(v, u)_M$ if we write u for v and conversely v for u .

The expressions $K_0(u, v)$ and $M_0(u, v)$ defined by the right sides of equations (7) are called boundary and continuity expressions.

Eigenvalue problem (1), (2) is said to be self-adjoint if the products (4) are commutative, i.e., if it holds that

$$(u, v)_K = (v, u)_K \quad \text{and} \quad (u, v)_M = (v, u)_M. \quad (8)$$

Conditions (8) are called conditions of self-adjointness.

2.3. Consequences. It follows from equations (7a) and (7b) that

$$(u, v)_K - (v, u)_K = K_0(u, v) - K_0(v, u) = 0 \quad (9a)$$

and

$$(u, v)_M - (v, u)_M = M_0(u, v) - M_0(v, u) = 0 \quad (9b)$$

if the eigenvalue problem (1), (2) is self-adjoint.

Let λ_ℓ be the ℓ -th eigenvalue ($\ell = 1, 2, 3, \dots$). The corresponding eigenfunctions (solutions to the eigenvalue problem (1), (2)) are denoted by y_ℓ .

Assume that the three point boundary value problem (1), (2) is self-adjoint. Then the eigenfunctions are orthogonal to each other in general sense:

$$(y_k, y_\ell)_K = \begin{cases} \lambda_n (y_k, y_\ell)_M & \text{if } k = \ell, \\ 0 & \text{if } k \neq \ell. \end{cases} \quad k, \ell = 1, 2, 3, \dots \quad (10)$$

In addition to this, the eigenvalues are all real numbers. These statements can be proved easily by recalling the similar proofs given for two point boundary value problems in [4].

2.4. Sign of eigenvalues. The three point eigenvalue problem (1), (2) is said to be positive definite if the eigenvalues are positive, positive semidefinite if one eigenvalue is zero and the other eigenvalues are all positive, negative semidefinite if one eigenvalue is zero and the other eigenvalues are all negative and finally negative definite if the eigenvalues are negative.

If we equalize ℓ and k in (10) we get

$$(y_\ell, y_\ell)_K = \lambda_k (y_\ell, y_\ell)_M \quad (11)$$

Hence

$$\lambda_k = \frac{(y_\ell, y_\ell)_K}{(y_\ell, y_\ell)_M} = \frac{\int_a^b y_\ell(x) K[y_\ell(x)] dx}{\int_a^b y_\ell(x) M[y_\ell(x)] dx} \quad (12)$$

This equation shows that the sign of λ_ℓ is a function of the products $(y_\ell, y_\ell)_K$ and $(y_\ell, y_\ell)_M$.

Assume that

$$(u, u)_K > 0, \quad \text{and} \quad (u, u)_M > 0 \quad (13)$$

for any comparison function $u(x)$. Then the three point eigenvalue problem (1), (2) is positive definite (or full definite).

The Raleigh quotient is defined by the equation

$$\mathcal{R}[u(x)] = \frac{(u, u)_K}{(u, u)_M} = \frac{\int_a^b u(x) K[u(x)] dx}{\int_a^b u(x) M[u(x)] dx} \quad (14)$$

in which $u(x)$ is a comparison function. Upon substitution of (7a) and (7b) for the numerator and denominator in (12) we get

$$\mathcal{R}[u(x)] = \frac{(u, u)_K}{(u, u)_M} = \frac{\sum_{n=0}^{\kappa} \int_a^b u^{(n)}(x) f_n(x) u^{(n)}(x) dx + K_0[u(x)]}{\sum_{n=0}^{\mu} \int_a^b u^{(n)}(x) g_n(x) u^{(n)}(x) dx + M_0[u(x)]} \quad (15)$$

If the three point eigenvalue problem (1), (2) is self-adjoint then

$$K_0[u(x)] = M_0[u(x)] = 0 \quad (16)$$

Assume that the three point eigenvalue problem (1), (2) is self adjoint and

$$M[y] = g_0(x)y(x) \quad (17)$$

Then the considered self-adjoint three point eigenvalue problem is simple.

2.5. Determination of the eigenvalues. Let us denote the linearly independent particular solutions of the differential equation $K[y] = \lambda M[y]$ by $z_\ell(x, \lambda)$ ($\ell = 1, 2, \dots, 2\kappa$). With $z_\ell(x, \lambda)$ the general solution is of the form

$$\begin{aligned} y_I(x) &= \sum_{\ell=1}^{2\kappa} \mathcal{A}_{\ell I} z_{\ell I}(x, \lambda) & \text{if } x \in [a, b] \\ y_{II}(x) &= \sum_{\ell=1}^{2\kappa} \mathcal{A}_{\ell II} z_{\ell II}(x, \lambda) & \text{if } x \in [b, c] \end{aligned} \quad (18)$$

where $z_\ell(x, \lambda) = z_{\ell I}(x, \lambda) = z_{\ell II}(x, \lambda)$. The undetermined integration constants $\mathcal{A}_{\ell I}$ and $\mathcal{A}_{\ell II}$ can be obtained from the boundary and continuity conditions:

$$\sum_{\ell=1}^{2\kappa} \mathcal{A}_{\ell I} U_{ar}[z_{\ell I}] = 0 \quad r = 1, 2, \dots, \kappa \quad (19a)$$

$$\sum_{\ell=1}^{2\kappa} \mathcal{A}_{\ell I} U_{brI}[z_{\ell I}] - \mathcal{A}_{\ell II} U_{brII}[z_{\ell II}] = 0 \quad r = 1, 2, \dots, 2\kappa \quad (19b)$$

$$\sum_{\ell=1}^{2\kappa} \mathcal{A}_{\ell II} U_{cr}[z_{\ell II}] = 0 \quad r = 1, 2, \dots, \kappa. \quad (19c)$$

Since these equations constitute a homogeneous linear equation system for the unknowns $\mathcal{A}_{\ell I}$ and $\mathcal{A}_{\ell II}$ solutions different from the trivial one (nontrivial solutions) exist if and only if the determinant of the system is zero:

$$\Delta(\lambda) = \begin{vmatrix} U_{ar}[z_{\ell I}] \\ U_{brI}[z_{\ell I}] - U_{brII}[z_{\ell II}] \\ U_{cr}[z_{\ell II}] \end{vmatrix} = 0. \quad (20)$$

This is the equation which should be solved to find the eigenvalues λ . We remark that the determinant $\Delta(\lambda)$ is referred to as characteristic determinant.

If $\Delta(\lambda)$ is identically equal to zero then each λ is an eigenvalue. Otherwise function $\Delta(\lambda)$ has an infinite sequence of isolated zero points which can be ordered according to their magnitudes:

$$0 \leq |\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \dots$$

3. THE GREEN FUNCTION

Consider the inhomogeneous ordinary differential equation

$$L[y(x)] = r(x) \quad (21a)$$

where the differential operator of order 2κ is defined by the following equation:

$$L[y(x)] = \sum_{n=0}^{2\kappa} p_n(x) y^{(n)}(x). \quad (21b)$$

Here $\kappa \geq 1$ is a natural number, the functions $p_n(x)$ and $r(x)$ are continuous if $x \in [a, c]$ ($c > a$, $c - a = \ell$) and $p_{2\kappa}(x) \neq 0$. Further let b be an inner point in the interval $[a, c]$: $b \in [a, b]$, $b - a = \ell_1$, $c - b = \ell_2$, $\ell_1 + \ell_2 = \ell$.

We shall assume that the inhomogeneous differential equation (21) is associated with the homogenous boundary and continuity conditions given by equation (2).

Solution of the three point boundary value problem (21), (2) is sought in the form

$$y(x) = \int_a^c G(x, \xi) r(\xi) d\xi \quad (22)$$

where $G(x, \xi)$ is the Green² function [1, 2, 3] defined by the following properties:

1. The Green function has the following structure:

$$G(x, \xi) = \begin{cases} G_{1I}(x, \xi) & \text{if } x, \xi \in [a, b] \\ G_{2I}(x, \xi) & \text{if } x \in [b, c] \text{ but } \xi \in [a, b] \\ G_{1II}(x, \xi) & \text{if } x \in [b, c] \text{ but } \xi \in [a, b] \\ G_{2II}(x, \xi) & \text{if } x, \xi \in [b, c]. \end{cases} \quad (23)$$

2. The function $G_{1I}(x, \xi)$ is a continuous function of x and ξ in the triangle $a \leq x \leq \xi \leq b$ and $a \leq \xi \leq x \leq b$. In addition, it is 2κ times differentiable with respect to x and the derivatives

$$\frac{\partial^n G_{1I}(x, \xi)}{\partial x^n} = G_{1I}(x, \xi)^{(n)}(x, \xi), \quad (n = 1, 2, \dots, 2\kappa)$$

are also continuous functions of x and ξ in the triangles $a \leq x \leq \xi \leq b$ and $a \leq \xi \leq x \leq b$.

3. Let ξ be fixed in $[a, b]$. The function $G_{1I}(x, \xi)$ and its derivatives

$$G_{1I}^{(n)}(x, \xi) = \frac{\partial^n G_{1I}(x, \xi)}{\partial x^n}, \quad (n = 1, 2, \dots, 2\kappa - 2) \quad (24)$$

²George Green (1793–1841)

should be continuous for $x = \xi$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{1I}^{(n)}(\xi + \varepsilon, \xi) - G_{1I}^{(n)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{1I}^{(n)}(\xi + 0, \xi) - G_{1I}^{(n)}(\xi - 0, \xi) \right] = 0 \quad n = 0, 1, 2, \dots, 2\kappa - 2. \end{aligned} \quad (25a)$$

The derivative $G_{1I}^{(2\kappa-1)}(x, \xi)$ should, however, have a jump

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{1I}^{(2\kappa-1)}(\xi + \varepsilon, \xi) - G_{1I}^{(2\kappa-1)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{1I}^{(2\kappa-1)}(\xi + 0, \xi) - G_{1I}^{(2\kappa-1)}(\xi - 0, \xi) \right] = \frac{1}{p_\kappa(\xi)} \end{aligned} \quad (25b)$$

if $x = \xi$.

In contrast to this, $G_{2I}(x, \xi)$ and its derivatives

$$G_{2I}^{(n)}(x, \xi) = \frac{\partial^n G_{2I}(x, \xi)}{\partial x^n}, \quad (n = 1, 2, \dots, 2\kappa) \quad (26)$$

are all continuous functions for any x in $[b, c]$.

4. Let ξ be fixed in $[b, c]$. The function $G_{1II}(x, \xi)$ and its derivatives

$$G_{1II}^{(n)}(x, \xi) = \frac{\partial^n G_{1II}(x, \xi)}{\partial x^n}, \quad (n = 1, 2, \dots, 2\kappa) \quad (27)$$

are all continuous functions for any x in $[a, c]$.

Though the function $G_{2II}(x, \xi)$ and its derivatives

$$G_{2II}^{(n)}(x, \xi) = \frac{\partial^n G_{2II}(x, \xi)}{\partial x^n}, \quad (n = 1, 2, \dots, 2\kappa - 2) \quad (28)$$

are also continuous for $x = \xi$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{2II}^{(n)}(\xi + \varepsilon, \xi) - G_{2II}^{(n)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{2II}^{(n)}(\xi + 0, \xi) - G_{2II}^{(n)}(\xi - 0, \xi) \right] = 0 \quad n = 0, 1, 2, \dots, 2\kappa - 2 \end{aligned} \quad (29a)$$

the derivative $G_{2II}^{(2\kappa-1)}(x, \xi)$ should, however, have a jump

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{2II}^{(2\kappa-1)}(\xi + \varepsilon, \xi) - G_{2II}^{(2\kappa-1)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{2II}^{(2\kappa-1)}(\xi + 0, \xi) - G_{2II}^{(2\kappa-1)}(\xi - 0, \xi) \right] = \frac{1}{p_\kappa(\xi)} \end{aligned} \quad (29b)$$

if $x = \xi$.

5. Let α be an arbitrary but finite nonzero constant. For a fixed $\xi \in [a, c]$ the product $G(x, \xi)\alpha$ as a function of x ($x \neq \xi$) should satisfy the homogeneous differential equation

$$L[G(x, \xi)\alpha] = 0.$$

6. The product $G(x, \xi)\alpha$ as a function of x should satisfy both the boundary conditions and the continuity conditions

$$\begin{aligned} U_{ar}[G] &= \sum_{n=1}^{2\kappa} \alpha_{nrI} G^{(n-1)}(a) = 0, & r &= 1, 2, \dots, \kappa \\ U_{br}[G] &= U_{brI}[G] - U_{brII}[G] = \\ &= \sum_{n=1}^{2\kappa} \left(\beta_{nrI} G^{(n-1)}(b-0) - \beta_{nrII} G^{(n-1)}(b+0) \right) = 0, & r &= 1, 2, \dots, 2\kappa \\ U_{cr}[G] &= \sum_{n=1}^{2\kappa} \gamma_{nrII} G^{(n-1)}(c) = 0, & r &= 1, 2, \dots, \kappa. \end{aligned} \quad (30)$$

These criteria should be applied to the function pairs $G_{1I}(x, \xi)$, $G_{2I}(x, \xi)$ and $G_{1II}(x, \xi)$, $G_{2II}(x, \xi)$ as well.

4. CALCULATION OF THE GREEN FUNCTIONS

4.1. Introductory remarks. Let us denote the linearly independent particular solutions of the homogeneous ordinary differential equation

$$L[y(x)] = 0 \quad (31)$$

by

$$z_1(x), z_2(x), z_3(x), \dots, z_\kappa(x). \quad (32)$$

Since the general solution is a linear combination of the particular solutions it can be given in the following form:

$$y(x) = \sum_{n=1}^{2\kappa} \mathcal{A}_n z_n(x) \quad (33)$$

were the coefficients \mathcal{A}_n are arbitrary integration constants. The Green function should satisfy the homogeneous differential equation (31). Therefore, it follows that it can be given as a linear combination of the particular solutions $z_n(x)$, i.e., by equation (33).

4.2. Calculation of the Green function if $\xi \in [a, b]$. It is obvious that the integration constants \mathcal{A}_n should be different in the two triangular domains $a \leq x \leq \xi \leq b$ and $a \leq \xi \leq x \leq b$. For this reason we shall apply the following assumptions:

(a)

$$\begin{aligned} G_{1I}(x, \xi) &= \sum_{\ell=1}^{2\kappa} (a_{\ell I}(\xi) + b_{\ell I}(\xi)) z_\ell(x), & x \leq \xi; \\ G_{1I}(x, \xi) &= \sum_{\ell=1}^{2\kappa} (a_{\ell I}(\xi) - b_{\ell I}(\xi)) z_\ell(x), & x \geq \xi \end{aligned} \quad x \in [a, b] \quad (34)$$

and

(b)

$$G_{2I}(x, \xi) = \sum_{\ell=1}^{2\kappa} c_{\ell I}(\xi) z_\ell(x), \quad x \in [b, c] \quad (35)$$

where the coefficients $a_{\ell I}(\xi)$, $b_{\ell I}(\xi)$ and $c_{\ell I}(\xi)$ are, in fact, unknown functions.

Continuity conditions (25a) yield the following equations

$$\sum_{\ell=1}^{2\kappa} b_{\ell I}(\xi) z_\ell^{(n)}(\xi) = 0, \quad n = 0, 1, 2, \dots, 2\kappa - 2. \quad (36a)$$

As regards discontinuity condition (25b) we get

$$\frac{1}{p_{2\kappa}(\xi)} = \left[G^{(2\kappa-1)}(\xi + 0, \xi) - G^{(2\kappa-1)}(\xi - 0, \xi) \right] = -2 \sum_{\ell=1}^{\kappa} b_{\ell I}(\xi) z_\ell^{(2\kappa-1)}(\xi).$$

Hence

$$\sum_{\ell=1}^{2\kappa} b_{\ell I}(\xi) z_\ell^{(2\kappa-1)}(\xi) = -\frac{1}{2p_{2\kappa}(\xi)}. \quad (36b)$$

Continuity and discontinuity conditions (46) have resulted in the inhomogeneous linear system of equations (36) for the unknowns $b_\ell(\xi)$ ($\ell = 1, 2, \dots, 2\kappa$). Its determinant assumes the following form:

$$\begin{vmatrix} z_1(\xi) & z_2(\xi) & \dots & z_\kappa(\xi) \\ z_1^{(1)}(\xi) & z_2^{(1)}(\xi) & \dots & z_\kappa^{(1)}(\xi) \\ \dots & \dots & \dots & \dots \\ z_1^{(2\kappa-1)}(\xi) & z_2^{(2\kappa-1)}(\xi) & \dots & z_{2\kappa}^{(2\kappa-1)}(\xi) \end{vmatrix}. \quad (37)$$

This determinant is the Wronskian³ [15] of differential equation (21) which is nonzero since the particular solutions $z_\ell(x)$ are linearly independent. This means that we can always find a unique solution for the coefficients $b_{\ell I}(\xi)$ in representation (34) of the Green function.

Utilizing the boundary and continuity conditions (30) the following equations can be obtained for the unknown coefficients $a_{\ell I}(\xi)$ and $c_{\ell I}(\xi)$:

$$U_{ar}[G] = \sum_{n=1}^{2\kappa} \alpha_{nrI} \sum_{\ell=1}^{2\kappa} (a_{\ell I}(\xi) + b_{\ell I}(\xi)) z_\ell(a)^{(n-1)} = 0 \quad r = 1, 2, \dots, \kappa \quad (38a)$$

³Józef Hoene-Wroński (1776-1853)

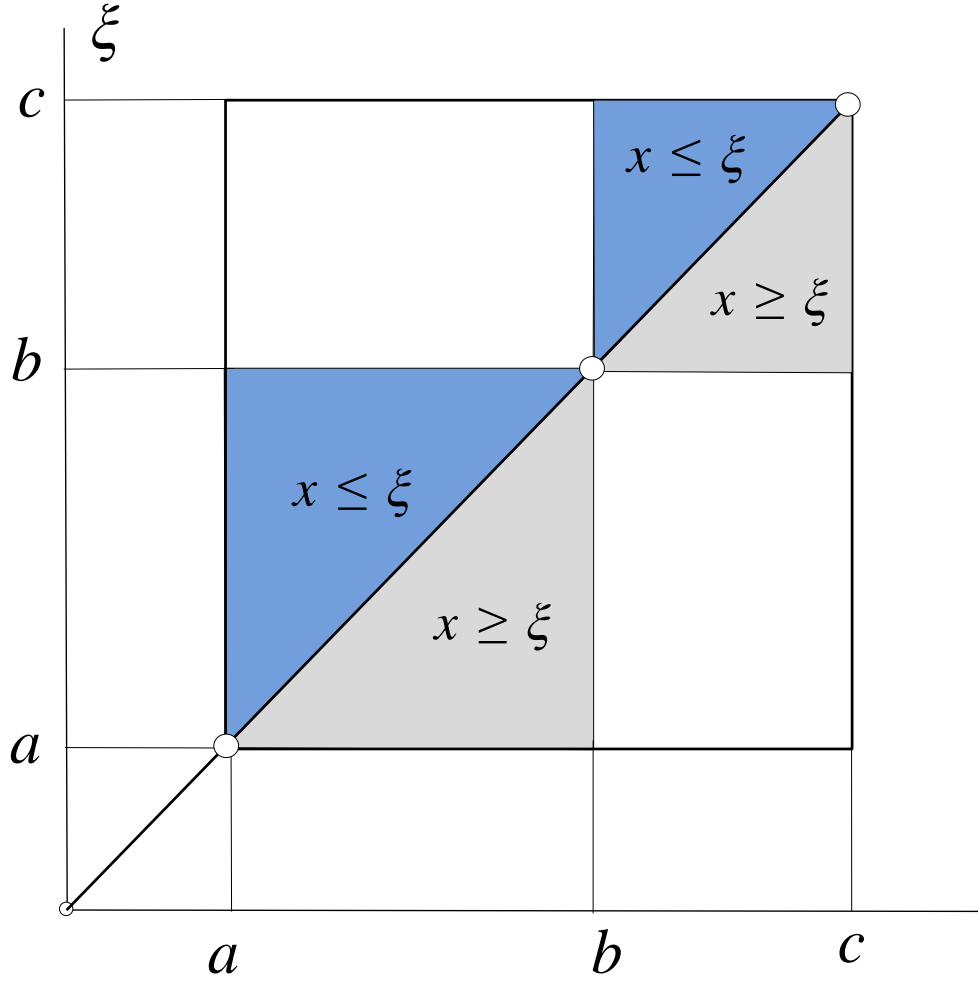


FIGURE 1. The domain considered

$$U_{br}[G] = \sum_{n=1}^{2\kappa} \left(\beta_{nrI} \sum_{\ell=1}^{2\kappa} (a_{\ell I}(\xi) - b_{\ell I}(\xi)) z_{\ell}(b)^{(n-1)} - \beta_{nrII} \sum_{\ell=1}^{2\kappa} c_{\ell}(\xi) z_{\ell}(b)^{(n-1)} \right) = 0, \quad r = 1, 2, \dots, 2\kappa \quad (38b)$$

$$U_{cr}[G] = \sum_{n=1}^{2\kappa} \gamma_{nrII} \sum_{\ell=1}^{2\kappa} c_{\ell I}(\xi) z_{\ell}(c)^{(n-1)} = 0, \quad r = 1, 2, \dots, \kappa \quad (38c)$$

or

$$\sum_{\ell=1}^{2\kappa} a_{\ell I} U_{ar}[z_{\ell}] = - \sum_{\ell=1}^{2\kappa} b_{\ell I} U_{ar}[z_{\ell}] \quad r = 1, 2, \dots, \kappa \quad (39a)$$

$$\sum_{\ell=1}^{2\kappa} a_{\ell I} U_{brI}[z_{\ell}] - \sum_{\ell=1}^{2\kappa} c_{\ell I} U_{brII}[z_{\ell}] = \sum_{\ell=1}^{2\kappa} b_{\ell I} U_{brI}[z_{\ell}] \quad r = 1, 2, \dots, 2\kappa \quad (39b)$$

$$\sum_{\ell=1}^{2\kappa} c_{\ell I} U_{cr}[z_{\ell}] = 0 \quad r = 1, 2, \dots, \kappa. \quad (39c)$$

Solutions for $a_{\ell I}(\xi)$ and $c_{\ell I}(\xi)$ exist if and only if the determinant of the linear equation system (39) is different from zero. The linear independence of the boundary and continuity conditions assures in general that the equation system (39) is solvable.

4.3. Calculation of the Green function if $\xi \in [b, c]$. The procedure that should be applied is similar to the procedure applied in the previous subsection. It is assumed that

(a)

$$\begin{aligned}
G_{2II}(x, \xi) &= \sum_{\ell=1}^{2\kappa} (a_{\ell II}(\xi) + b_{\ell II}(\xi)) z_{\ell}(x), & x \leq \xi; \\
G_{1II}(x, \xi) &= \sum_{\ell=1}^{2\kappa} (a_{\ell II}(\xi) - b_{\ell II}(\xi)) z_{\ell}(x), & x \geq \xi
\end{aligned} \quad x \in [b, c] \quad (40)$$

and

(b)

$$G_{1II}(x, \xi) = \sum_{\ell=1}^{2\kappa} c_{\ell II}(\xi) z_{\ell}(x), \quad x \in [a, b] \quad (41)$$

where the coefficients $a_{\ell II}(\xi)$, $b_{\ell II}(\xi)$ and $c_{\ell II}(\xi)$ are again unknown functions.

Continuity conditions (29a) lead to the following equations:

$$\sum_{\ell=1}^{2\kappa} b_{\ell II}(\xi) z_{\ell}^{(n)}(\xi) = 0, \quad n = 0, 1, 2, \dots, 2\kappa - 2. \quad (42a)$$

As regards discontinuity condition (29b) we obtain

$$\frac{1}{p_{2\kappa}(\xi)} = \left[G^{(2\kappa-1)}(\xi + 0, \xi) - G^{(2\kappa-1)}(\xi - 0, \xi) \right] = -2 \sum_{\ell=1}^{\kappa} b_{\ell II}(\xi) z_{\ell}^{(2\kappa-1)}(\xi).$$

Hence

$$\sum_{\ell=1}^{2\kappa} b_{\ell II}(\xi) z_{\ell}^{(2\kappa-1)}(\xi) = -\frac{1}{2p_{2\kappa}(\xi)}. \quad (42b)$$

Continuity and discontinuity conditions (29) have yielded the inhomogeneous linear system of equations (42) for the unknowns $b_{\ell II}(\xi)$ ($\ell = 1, 2, \dots, 2\kappa$). Since these equations are formally the same as equations (36) it follows that their determinant is again the Wronskian (37) and this fact guarantees their solvability.

Utilizing now the boundary and continuity conditions (30) leads to the following linear equations for the unknown coefficients $a_{\ell II}(\xi)$ and $c_{\ell II}(\xi)$:

$$U_{ar}[G] = \sum_{n=1}^{2\kappa} \alpha_{nrI} \sum_{\ell=1}^{2\kappa} c_{\ell II}(\xi) z_{\ell}(a)^{(n-1)} = 0 \quad r = 1, 2, \dots, \kappa \quad (43a)$$

$$\begin{aligned}
U_{br}[G] &= \sum_{n=1}^{2\kappa} \left(\beta_{nrI} \sum_{\ell=1}^{2\kappa} c_{\ell II}(\xi) z_{\ell}(b)^{(n-1)} - \right. \\
&\quad \left. - \beta_{nrII} \sum_{\ell=1}^{2\kappa} (a_{\ell II}(\xi) + b_{\ell II}(\xi)) z_{\ell}(b)^{(n-1)} \right) = 0, \quad r = 1, 2, \dots, 2\kappa \quad (43b)
\end{aligned}$$

$$U_{cr}[G] = \sum_{n=1}^{2\kappa} \gamma_{nrII} \sum_{\ell=1}^{2\kappa} (a_{\ell II}(\xi) - b_{\ell II}(\xi)) z_{\ell}(c)^{(n-1)} = 0, \quad r = 1, 2, \dots, \kappa \quad (43c)$$

or

$$\sum_{\ell=1}^{2\kappa} a_{\ell II} U_{ar}[z_{\ell}] = 0 \quad r = 1, 2, \dots, \kappa \quad (44a)$$

$$\sum_{\ell=1}^{2\kappa} a_{\ell II} U_{brI}[z_{\ell}] - \sum_{\ell=1}^{2\kappa} b_{\ell II} U_{brII}[z_{\ell}] = \sum_{\ell=1}^{2\kappa} c_{\ell II} U_{brII}[z_{\ell}] \quad r = 1, 2, \dots, 2\kappa \quad (44b)$$

$$\sum_{\ell=1}^{2\kappa} b_{\ell II} U_{cr}[z_{\ell}] = \sum_{\ell=1}^{2\kappa} c_{\ell II} U_{cr}[z_{\ell}] \quad r = 1, 2, \dots, \kappa \quad (44c)$$

The linear equation system (43) is solvable for $a_{\ell II}(\xi)$ and $b_{\ell II}(\xi)$ if its determinant is different from zero. Since the boundary and continuity conditions are linearly independent equation system (43) is, in general, solvable.

If the determinants of the equation systems (39) and (44) are different from zero then there exists a Green function which satisfies the properties of the definition given in Section 3.

5. SOME PROPERTIES OF THE GREEN FUNCTION

If there exists the Green function defined in Section 3 for the three point boundary value problem (21), (2) then the function $y(x)$ given by (22) satisfies differential equation (21) and, in addition to this, the boundary and the continuity conditions (2) as well.

Using (22) we can determine the first $\kappa - 1$ derivatives of $y(x)$:

$$\begin{aligned} y^{(1)}(x) &= \int_a^c G^{(1)}(x, \xi)r(\xi) d\xi, & y^{(2)}(x) &= \int_a^b G^{(2)}(x, \xi)r(\xi) d\xi, \dots \\ \dots, y^{(\kappa-1)}(x) &= \int_a^c G^{(\kappa-1)}(x, \xi)r(\xi) d\xi = \int_a^x G^{(\kappa-1)}(x, \xi)r(\xi) d\xi + \int_x^c G^{(\kappa-1)}(x, \xi)r(\xi) d\xi. \end{aligned} \quad (45)$$

When calculating the κ^{th} derivative we have to take, however, into account the additive resolution of the $(\kappa - 1)^{\text{th}}$ derivative as well as discontinuities (25b) and (29b). We get

$$\begin{aligned} y^{(\kappa)}(x) &= \int_a^c G^{(\kappa)}(x, \xi)r(\xi) d\xi + G^{(\kappa-1)}(x, \xi)r(\xi) \Big|_{\xi=x-0} - G^{(\kappa-1)}(x, \xi)r(\xi) \Big|_{\xi=x+0} = \\ &= \int_a^c G^{(\kappa)}(x, \xi)r(\xi) d\xi + \left[G^{(\kappa-1)}(x, x-0) - G^{(\kappa-1)}(x, x+0) \right] r(x) \end{aligned}$$

where $x > x - 0$, $x < x + 0$ and $x \in (a, b)$ or $x \in (b, c)$. Hence

$$y^{(\kappa)}(x) = \int_a^c G^{(\kappa)}(x, \xi)r(\xi) d\xi + \frac{r(x)}{p_\kappa(x)}. \quad (46)$$

If we substitute solution (22) and derivatives (45), (46) back into the differential equation (21) we get

$$L[y(x)] = \sum_{n=0}^{\kappa} p_n(x) y^{(n)}(x) = \int_a^b \left\{ \sum_{n=0}^{\kappa} p_n(x) G^{(n)}(x, \xi) \right\} r(\xi) d\xi + \frac{r(x)}{p_\kappa(x)} p_\kappa(x) = r(x) \quad (47)$$

where the expression within the braces is zero due to Property 5 of the definition – Properties 1, 2, 3 and 4 were all taken into account when we determined the derivatives of the Green function. Equation (47) shows that integral (22) is really a solution of differential equation (21). According to Property 6 the Green function satisfies the boundary and continuity conditions. Hence integral (22) is really the solution of the boundary value problem (21),(2) as well.

Consider now the following two inhomogeneous three point boundary value problems:

$$L[u(x)] = r(x), \quad U_r[u(x)] = 0; \quad (48)$$

$$L[v(x)] = s(x), \quad U_r[v(x)] = 0 \quad (49)$$

where $u(x)$ and $v(x)$ are the unknown functions while $r(x)$ and $s(x)$ are continuous inhomogeneities in the interval $x \in [a, b]$. We shall assume that boundary value problems (48) and (49) are self-adjoint. Then

$$(u, v)_L - (v, u)_L = 0$$

in which according to equation (22)

$$u(x) = \int_a^b G(x, \xi)r(\xi) d\xi \quad \text{and} \quad v(x) = \int_a^b G(x, \xi)s(\xi) d\xi.$$

Thus

$$\begin{aligned} (u, v)_L - (v, u)_L &= \int_a^b (u L[v] - v L[u]) dx = \\ &= \int_a^b \int_a^b [G(x, \xi) - G(\xi, x)] r(\xi)s(x) d\xi dx = 0. \end{aligned} \quad (50)$$

Since both $r(x)$ and $s(x)$ are arbitrary continuous and nonzero functions in the interval $[a, b]$ it follows that the last integral in equation (50) can be zero if and only if

$$\boxed{G(x, \xi) = G(\xi, x)}. \quad (51)$$

In words: the Green function of self-adjoint three point boundary value problems is a symmetric function.

6. EXAMPLES

6.1. Preliminary remarks. The examples presented in this section are all related to a class of heterogeneous beams. The centerline of the beam coincides with the coordinate axis x . The plane of the cross sections is therefore perpendicular to the axis x . The coordinate plane xz is a plane of symmetry. The material of the beam is isotropic but heterogeneous, i.e., Young's modulus is a function of the coordinates y, z : $E = E(y, z)$, $E(y, z) = E(-y, z)$.

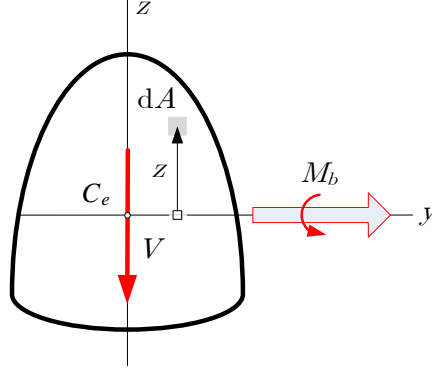


FIGURE 2. The investigated cross section

This heterogeneity is called cross sectional heterogeneity [16]. The E weighted cross sectional area, first moment, and moment of inertia are defined by the following integrals:

$$A_e = \int_A E(y, z) dA, \quad Q_{ey} = \int_A E(y, z) z dA, \quad I_{ey} = \int_A E(y, z) z^2 dA.$$

The location of the E -weighted center C_e is determined by the condition $Q_{ey} = 0$. We remark that Figure 2 shows the directions of the positive bending moment M_b and shear force V .

Equilibrium problems of a uniform heterogeneous beam subjected to an axial force N are governed by the differential equation

$$\frac{d^4 w}{dx^4} \pm \frac{N}{I_{ey}} \frac{d^2 w}{dx^2} = \frac{f_z}{I_{ey}} \quad (52)$$

where f_z is the intensity of the distributed load acting on the axis x and the axial force N is constant ($N > 0$) while its sign in the above equation is positive if the axial force is compressive and negative if the force is tensile. If the beam is homogenous $I_{ey} = I_y E$ where E is constant and

$$I_y = \int_A z^2 dA.$$

6.2. Three point boundary value problems for beams.

6.2.1. Problem 1. If there is no axial force ($N = 0$) equilibrium problems of heterogeneous uniform beams are governed by the simple differential equation

$$L(w) = \frac{d^4 w}{dx^4} = \frac{f_z(x)}{I_{ye}} = \hat{f}(x). \quad (53)$$

For the beam shown in Figure 2 this equation is

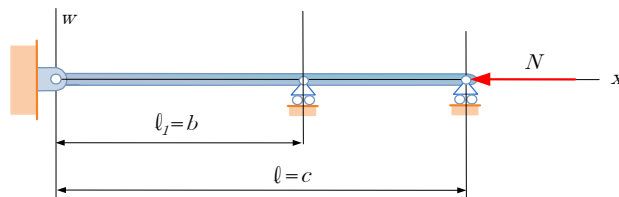


FIGURE 3. The beam centerline with the load and supports

associated with the following boundary and continuity conditions:

$$\begin{aligned} w(0) &= 0, & w^{(2)}(0) &= 0, \\ w(\ell) &= 0, & w^{(2)}(\ell) &= 0, \end{aligned} \quad (54a)$$

$$\begin{aligned}
w(b-0) &= 0, & w(b+0) &= 0, \\
w^{(1)}(b-0) &= w^{(1)}(b+0), \\
w^{(2)}(b-0) &= w^{(2)}(b+0).
\end{aligned} \tag{54b}$$

The three point boundary value problem (53), (54) is self-adjoint. Our aim is to find the Green function.

The linearly independent particular solutions of equation $L(w) = 0$ are given by

$$w_1 = 1, \quad w_2 = x, \quad w_3 = x^2, \quad w_4 = x^3. \tag{55}$$

Calculation of the Green function if $\xi \in (a = 0, b)$:

The continuity and discontinuity conditions (36) result in the following equation system for $b_{\ell I}$:

$$\begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \\ 0 & 1 & 2\xi & 3\xi^2 \\ 0 & 0 & 2 & 6\xi \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

from where

$$\begin{bmatrix} b_{1I} \\ b_{2I} \\ b_{3I} \\ b_{4I} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} \xi^3 \\ -3\xi^2 \\ 3\xi \\ -1 \end{bmatrix}. \tag{56}$$

Boundary conditions (54a) and continuity conditions (54b) lead to the following equation system:

(a) Boundary conditions at $x = 0$:

$$\begin{aligned}
a_{1I}w_1(0) + a_{2I}w_2(0) + a_{3I}w_3(0) + a_{4I}w_4(0) &= \\
&= -b_{1I}w_1(0) - b_{2I}w_2(0) - b_{3I}w_3(0) - b_{4I}w_4(0), \tag{57a}
\end{aligned}$$

$$\begin{aligned}
a_{1I}w_1^{(2)}(0) + a_{2I}w_2^{(2)}(0) + a_{3I}w_3^{(2)}(0) + a_{4I}w_4^{(2)}(0) &= \\
&= -b_{1I}w_1^{(2)}(0) - b_{2I}w_2^{(2)}(0) - b_{3I}w_3^{(2)}(0) - b_{4I}w_4^{(2)}(0). \tag{57b}
\end{aligned}$$

(b) Continuity conditions at $x = b$:

$$\begin{aligned}
a_{1I}w_1(b) + a_{2I}w_2(b) + a_{3I}w_3(b) + a_{4I}w_4(b) &= \\
&= b_{1I}w_1(b) + b_{2I}w_2(b) + b_{3I}w_3(b) + b_{4I}w_4(b), \tag{57c}
\end{aligned}$$

$$c_{1I}w_1(b) + c_{2I}w_2(b) + c_{3I}w_3(b) + c_{4I}w_4(b) = 0, \tag{57d}$$

$$\begin{aligned}
a_{1I}w_1^{(1)}(b) + a_{2I}w_2^{(1)}(b) + a_{3I}w_3^{(1)}(b) + a_{4I}w_4^{(1)}(b) - \\
- c_{1I}w_1^{(1)}(b) - c_{2I}w_2^{(1)}(b) - c_{3I}w_3^{(1)}(b) - c_{4I}w_4^{(1)}(b) &= \\
&= b_{1I}w_1^{(1)}(b) + b_{2I}w_2^{(1)}(b) + b_{3I}w_3^{(1)}(b) + b_{4I}w_4^{(1)}(b), \tag{57e}
\end{aligned}$$

$$\begin{aligned}
a_{1I}w_1^{(2)}(b) + a_{2I}w_2^{(2)}(b) + a_{3I}w_3^{(2)}(b) + a_{4I}w_4^{(2)}(b) - \\
- c_{1I}w_1^{(2)}(b) - c_{2I}w_2^{(2)}(b) - c_{3I}w_3^{(2)}(b) - c_{4I}w_4^{(2)}(b) &= \\
&= b_{1I}w_1^{(2)}(b) + b_{2I}w_2^{(2)}(b) + b_{3I}w_3^{(2)}(b) + b_{4I}w_4^{(2)}(b). \tag{57f}
\end{aligned}$$

(c) Boundary conditions at $x = \ell$:

$$c_{1I}w_1(\ell) + c_{2I}w_2(\ell) + c_{3I}w_3(\ell) + c_{4I}w_4(\ell) = 0, \tag{57g}$$

$$c_{1I}w_1^{(2)}(\ell) + c_{2I}w_2^{(2)}(\ell) + c_{3I}w_3^{(2)}(\ell) + c_{4I}w_4^{(2)}(\ell) = 0. \tag{57h}$$

In what follows we shall use the solutions for w and b_{1I}, \dots, b_{4I} without referring to them.

The boundary conditions at $x = 0$ yield

$$a_{1I} = -\frac{\xi^3}{12}, \quad a_{3I} = -\frac{3}{12}\xi. \tag{58}$$

As regards the continuity conditions we get

$$a_{1I} + a_{2I}b + a_{3I}b^2 + a_{4I}b^3 = \frac{1}{12}(\xi^3 - 3\xi^2b + 3\xi b^2 - b^3), \tag{59a}$$

$$c_{1I} + c_{2I}b + c_{3I}b^2 + c_{4I}b^3 = 0, \tag{59b}$$

$$a_{2I} + 2a_{3I}b + 3a_{4I}b^2 - c_{2I} - 2c_{3I}b - 3c_{4I}b^2 = \frac{1}{12} (-3\xi^2 + 6\xi b - 3b^2), \quad (59c)$$

$$2a_{3I} + 6a_{4I}b - 2c_{3I} - 6c_{4I}b = \frac{1}{12} (6\xi - 6b). \quad (59d)$$

The boundary conditions at the left end of the beam are of the form

$$c_{1I} + c_{2I}\ell + c_{3I}\ell^2 + c_{4I}\ell^3 = 0, \quad (60a)$$

$$2c_{3I} + 6c_{4I}\ell = 0. \quad (60b)$$

After eliminating a_{1I} and a_{3I} in equations (59) and (60) we arrive at the following matrix equation

$$\begin{bmatrix} b & b^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b & b^2 & b^3 \\ 1 & 3b^2 & 0 & -1 & -2b & -3b^2 \\ 0 & 6b & 0 & 0 & -2 & -6b \\ 0 & 0 & 1 & \ell & \ell^2 & \ell^3 \\ 0 & 0 & 0 & 0 & 2 & 6\ell \end{bmatrix} \begin{bmatrix} a_{2I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4I} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2\xi^3 - 3\xi^2b + 6\xi b^2 - b^3 \\ 0 \\ -3\xi^2 + 12\xi b - b^2 \\ 12\xi - 6b \\ 0 \\ 0 \end{bmatrix}. \quad (61)$$

The solutions are given by:

$$\begin{aligned} a_{1I} &= -\frac{1}{12}\xi^3 \\ a_{2I} &= \frac{1}{12b}\frac{\xi}{\ell} (-b^3 + 4lb^2 + b\xi^2 - 3lb\xi + 2l\xi^2) \\ a_{3I} &= -\frac{3}{12}\xi \\ a_{4I} &= -\frac{1}{24b^2\ell} (-5b^2\xi + 2lb^2 + 3b\xi^2 - 4lb\xi + 2\xi^3) \end{aligned} \quad (62)$$

and

$$\begin{aligned} c_{1I} &= \frac{1}{12}\frac{\xi}{(\ell - b)} (\xi^2 - b^2) (b - 2\ell) \\ c_{2I} &= \frac{\xi}{12b\ell(\ell - b)} (\xi^2 - b^2) (2\ell^2 - b^2 + 2b\ell) \\ c_{3I} &= -\frac{\xi}{4b(\ell - b)} (\xi^2 - b^2) \\ c_{4I} &= \frac{\xi}{12b\ell(\ell - b)} (\xi^2 - b^2). \end{aligned} \quad (63)$$

Hence

$$\begin{aligned} G_{1I}(x, \xi) &= \sum_{\ell=1}^4 (a_{\ell I}(\xi) \pm b_{\ell I}(\xi)) w_{\ell}(x) = \\ &= \left(-\frac{1}{12}\xi^3 \pm \frac{1}{12}\xi^3 \right) + \left(\frac{1}{12b}\frac{\xi}{\ell} (-b^3 + 4lb^2 + b\xi^2 - 3lb\xi + 2l\xi^2) \pm \frac{-3\xi^2}{12} \right) x + \\ &\quad + \left(-\frac{3\xi}{12} \pm \frac{3\xi}{12} \right) x^2 + \left(-\frac{1}{12b^2\ell} (-b^2\xi + lb^2 - 2lb\xi + \xi^3) \pm \frac{-1}{12} \right) x^3 \end{aligned} \quad (64)$$

and

$$\begin{aligned} G_{2I}(x, \xi) &= \sum_{\ell=1}^4 c_{\ell I}(\xi) w_{\ell}(x) = \\ &= \frac{1}{12b\ell(\ell - b)} \xi (b + \xi) (x - \ell) (b - x) (b + x - 2\ell) (b - \xi). \end{aligned} \quad (65)$$

Calculation of the Green function if $\xi \in (b, c = \ell)$:

The equation system obtained from the continuity and discontinuity conditions (42) coincides with the equation system set up for $b_{\ell I}$. Hence

$$b_{1II} = b_{1I}, \quad b_{2II} = b_{2I}, \quad b_{3II} = b_{3I}, \quad b_{4II} = b_{4I}. \quad (66)$$

Boundary conditions (54a) and continuity conditions (54b) lead to the following equation system:

(a) Boundary conditions at $x = 0$:

$$c_{1II}w_1(0) + c_{2II}w_2(0) + c_{3II}w_3(0) + c_{4II}w_4(0) = 0, \quad (67a)$$

$$c_{1II}w_1^{(2)}(0) + c_{2II}w_2^{(2)}(0) + c_{3II}w_3^{(2)}(0) + c_{4II}w_4^{(2)}(0) = 0. \quad (67b)$$

(b) Continuity conditions at $x = b$:

$$c_{1II}w_1(b) + c_{2II}w_2(b) + c_{3II}w_3(b) + c_{4II}w_4(b) = 0, \quad (67c)$$

$$\begin{aligned} a_{1II}w_1(b) + a_{2II}w_2(b) + a_{3II}w_3(b) + a_{4II}w_4(b) = \\ = -b_{1II}w_1(b) - b_{2II}w_2(b) - b_{3II}w_3(b) - b_{4II}w_4(b), \end{aligned} \quad (67d)$$

$$\begin{aligned} a_{1II}w_1^{(1)}(b) + a_{2II}w_2^{(1)}(b) + a_{3II}w_3^{(1)}(b) + a_{4II}w_4^{(1)}(b) - \\ - c_{1II}w_1^{(1)}(b) - c_{2II}w_2^{(1)}(b) - c_{3II}w_3^{(1)}(b) - c_{4II}w_4^{(1)}(b) = \\ = -b_{1II}w_1^{(1)}(b) - b_{2II}w_2^{(1)}(b) - b_{3II}w_3^{(1)}(b) - b_{4II}w_4^{(1)}(b), \end{aligned} \quad (67e)$$

$$\begin{aligned} a_{1II}w_1^{(2)}(b) + a_{2II}w_2^{(2)}(b) + a_{3II}w_3^{(2)}(b) + a_{4II}w_4^{(2)}(b) - \\ - c_{1II}w_1^{(2)}(b) - c_{2II}w_2^{(2)}(b) - c_{3II}w_3^{(2)}(b) - c_{4II}w_4^{(2)}(b) = \\ = -b_{1II}w_1^{(2)}(b) - b_{2II}w_2^{(2)}(b) - b_{3II}w_3^{(2)}(b) - b_{4II}w_4^{(2)}(b). \end{aligned} \quad (67f)$$

(c) Boundary conditions at $x = \ell$:

$$\begin{aligned} a_{1II}w_1(\ell) + a_{2II}w_2(\ell) + a_{3II}w_3(\ell) + a_{4II}w_4(\ell) - \\ - b_{1II}w_1(\ell) - b_{2II}w_2(\ell) - b_{3II}w_3(\ell) - b_{4II}w_4(\ell) = 0, \end{aligned} \quad (67g)$$

$$\begin{aligned} a_{1II}w_1^{(2)}(\ell) + a_{2II}w_2^{(2)}(\ell) + a_{3II}w_3^{(2)}(\ell) + a_{4II}w_4^{(2)}(\ell) - \\ - b_{1II}w_1^{(2)}(\ell) - b_{2II}w_2^{(2)}(\ell) - b_{3II}w_3^{(2)}(\ell) - b_{4II}w_4^{(2)}(\ell) = 0. \end{aligned} \quad (67h)$$

The boundary conditions at $x = 0$ yield

$$c_{1II} = c_{3II} = 0. \quad (68)$$

As regards the continuity conditions we get

$$c_{2II}b + c_{4II}b^3 = 0, \quad (69a)$$

$$a_{1II} + a_{2II}b + a_{3II}b^2 + a_{4II}b^3 = -\frac{1}{12}(\xi^3 - 3b\xi^2 + 3b^2\xi - b^3), \quad (69b)$$

$$a_{2II} + 2a_{3II}b + 3a_{4II}b^2 - c_{2II} - 3c_{4II}b^2 = -\frac{1}{12}(-3\xi^2 + 6b\xi - 3b^2), \quad (69c)$$

$$2a_{3II} + 6a_{4II}b - 6c_{4II}b = -\frac{1}{12}(6\xi + 6b). \quad (69d)$$

The boundary conditions at the left end of the beam are as follows

$$a_{1II} + a_{2II}\ell + a_{3II}\ell^2 + a_{4II}\ell^3 = \frac{1}{12}(\xi^3 - 3\xi^2\ell + 3\xi\ell - \ell^3), \quad (70a)$$

$$2a_{3II} + 6a_{4II}\ell = \frac{1}{12}(3\xi - 6\ell). \quad (70b)$$

After grouping equations (69) and (71) we get the following matrix equation

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b & b^3 \\ 1 & b & b^2 & b^3 & 0 & 0 \\ 0 & 1 & 2b & 3b^2 & -1 & -3b^2 \\ 0 & 0 & 2 & 6b & 0 & -6b \\ 1 & \ell & \ell^2 & \ell^3 & 0 & 0 \\ 0 & 0 & 2 & 6\ell & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4II} \\ c_{2II} \\ c_{4II} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 0 \\ -\xi^3 + 3b\xi^2 - 3b^2\xi + b^3 \\ 3\xi^2 - 6b\xi + 3b^2 \\ -6\xi + 6b \\ \xi^3 - 3\xi^2\ell + 3\xi\ell^2 - \ell^3 \\ 6\xi - 6\ell \end{bmatrix}. \quad (71)$$

Making use of the solutions

$$\begin{aligned} a_{1II} &= -\frac{1}{12(b-\ell)^2}(-b^4\xi + b^4\ell + 3b^2\xi^2\ell - 2b^2\xi\ell^2 - 2b\xi^3\ell + \xi^3\ell^2) \\ a_{2II} &= \frac{1}{12\ell(b-\ell)^2}(-b^4\xi + b^4\ell + b^2\xi^3 + 2b^2\ell^3 - \\ &\quad - 4b\xi^3\ell + 6b\xi^2\ell^2 - 8b\xi\ell^3 + 3\xi^2\ell^3) \\ a_{3II} &= \frac{1}{4} \frac{\xi}{(b-\ell)^2}(\xi^2 - 3\xi\ell + \ell^2 + 2b\ell) \\ a_{4II} &= -\frac{1}{12\ell(b-\ell)^2}(-b^2\xi + 4b\xi\ell - 2b\ell^2 + \xi^3 - 3\xi^2\ell + \ell^3) \end{aligned} \quad (72)$$

and

$$\begin{aligned} c_{2II} &= -\frac{1}{12} \frac{b}{\ell} \frac{b-\xi}{b-\ell} (\xi-\ell)(b+\xi-2\ell) \\ c_{4II} &= \frac{1}{12b\ell} \frac{b-\xi}{b-\ell} (\xi-\ell)(b+\xi-2\ell) \end{aligned} \quad (73)$$

we have

$$\begin{aligned} G_{2II}(x, \xi) &= \sum_{\ell=1}^4 (a_{\ell II}(\xi) \pm b_{\ell II}(\xi)) z_{\ell}(x) = \\ &= -\frac{1}{12(b-\ell)^2} (-b^4\xi + b^4\ell + 3b^2\xi^2\ell - 2b^2\xi\ell^2 - 2b\xi^3\ell + \xi^3\ell^2) \pm \frac{\xi^3}{12} + \\ &+ \left(\frac{1}{12\ell(b-\ell)^2} (-b^4\xi + b^4\ell + b^2\xi^3 + 2b^2\ell^3 - 4b\xi^3\ell + 6b\xi^2\ell^2 - 8b\xi\ell^3 + 3\xi^2\ell^3) \pm \frac{-3\xi^2}{12} \right) x + \\ &+ \left(\frac{1}{4(b-\ell)^2} (-b^2\ell + 2b\xi\ell + \xi^3 - 3\xi^2\ell + \xi\ell^2) \pm \frac{3\xi}{12} \right) x^2 + \\ &+ \left(-\frac{1}{12\ell(b-\ell)^2} (-b^2\xi + 4b\xi\ell - 2b\ell^2 + \xi^3 - 3\xi^2\ell + \ell^3) \pm \frac{-1}{12} \right) x^3 \end{aligned} \quad (74)$$

and

$$\begin{aligned} G_{1II}(x, \xi) &= \sum_{\ell=1}^4 c_{\ell II}(\xi) z_{\ell}(x) = \\ &= -\frac{1}{12} \frac{b+x}{b\ell(b-\ell)} x (\xi-\ell)(b-\xi)(b+\xi-2\ell)(b-x). \end{aligned} \quad (75)$$

6.2.2. Problem 2. If the axial force is nonzero ($N \neq 0$) but a compressive one the equilibrium problems of beams are governed by differential equation

$$L(w) = \frac{d^4w}{dx^4} + \mathcal{N} \frac{d^2w}{dx^2} = \hat{f}_z; \quad \mathcal{N} = \frac{N}{I_{ey}}. \quad (76)$$

For the beam shown in Figure 2 this equation is associated with the boundary and continuity conditions given by equations (54). Our aim is to find the Green function.

Making use of the linearly independent particular solutions

$$w_1 = 1, \quad w_2 = x, \quad w_3 = \cos px, \quad w_4 = \sin px, \quad p = \sqrt{\frac{N}{I_{ey}}}, \quad N > 0. \quad (77)$$

we can present the general solution in the form

$$\begin{aligned} w &= a_1 + a_2x + a_3 \cos px + a_4 \sin px \\ \frac{dw}{dx} &= a_2 - pa_3 \sin px + pa_4 \cos px, \\ \frac{d^2w}{dx^2} &= -p^2 a_3 \cos px - p^2 a_4 \sin px, \\ \frac{d^3w}{dx^3} &= p^3 a_3 \sin px - p^3 a_4 \cos px \end{aligned} \quad (78)$$

where a_1, \dots, a_4 are undetermined integration constants. Note that the derivatives used in the calculations are also given here.

Calculation of the Green function if $\xi \in (a=0, b)$:

The continuity and discontinuity conditions (36) result in the following equation system for $b_{\ell I}$:

$$\begin{bmatrix} 1 & \xi & \cos p\xi & \sin p\xi \\ 0 & 1 & -p \sin p\xi & p \cos p\xi \\ 0 & 0 & -p^2 \cos p\xi & -p^2 \sin p\xi \\ 0 & 0 & p^3 \sin p\xi & -p^3 \cos p\xi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \quad (79)$$

from where we get

$$\begin{bmatrix} b_{1I} \\ b_{2I} \\ b_{3I} \\ b_{4I} \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} p\xi \\ -p \\ -\sin p\xi \\ \cos p\xi \end{bmatrix}. \quad (80)$$

The boundary conditions at $x = 0$ lead to the following results

$$a_{1I} = -\frac{1}{2p^2}\xi, \quad a_{3I} = \frac{\sin p\xi}{2p^3}.$$

In what follows the details are omitted. The line of thought is the same as in the case of Problem 1. Continuity conditions (54b) and boundary conditions (54a)₂ yield the following equation system:

$$\begin{bmatrix} b & \sin pb & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b & \cos pb & \sin pb \\ 1 & p \cos pb & 0 & -1 & p \sin pb & -p \cos pb \\ 0 & -\sin pb & 0 & 0 & \cos pb & \sin pb \\ 0 & 0 & 1 & \ell & \cos p\ell & \sin p\ell \\ 0 & 0 & 0 & 0 & \cos p\ell & \sin p\ell \end{bmatrix} \begin{bmatrix} a_{2I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4I} \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} 2p\xi - pb - \sin p\xi \cos pb - \sin(p\xi - bp) \\ 0 \\ -p + 2p \sin p\xi \sin pb + p \cos p\xi \cos pb \\ 2 \sin p\xi \cos pb - \cos p\xi \sin pb \\ 0 \\ 0 \end{bmatrix}. \quad (81)$$

The solutions are given by

$$a_{1I} = -\frac{1}{2p^2}\xi, \quad (82a)$$

$$a_{2I} = \frac{1}{2p^3\mathcal{D}} [(\sin(p\ell - bp) \sin bp + p(\sin p\ell)(b - \ell))(\sin(p\xi - bp) - 2p\xi + bp + \sin p\xi \cos bp) + (\sin(p\ell - bp) \sin bp + p(\cos(p\ell - bp) \sin bp)(b - \ell))(\cos p\xi \sin bp - 2 \sin p\xi \cos bp) + (\sin(p\ell - bp) \sin bp)(b - \ell)(p \cos p\xi \cos bp - p + 2p \sin p\xi \sin bp)], \quad (82b)$$

$$a_{3I} = \frac{\sin p\xi}{2p^3}, \quad (82c)$$

$$a_{4I} = \frac{1}{2p^3\mathcal{D}} [-(\cos p\xi \sin bp - 2 \sin p\xi \cos bp)(b \sin(p\ell - bp) + bp(\cos(p\ell - bp))(b - \ell)) - (\sin(p\ell - bp))(b - \ell)(\sin(p\xi - bp) - 2p\xi + bp + \sin p\xi \cos bp) - b(\sin(p\ell - bp))(b - \ell)(p \cos p\xi \cos bp - p + 2p \sin p\xi \sin bp)], \quad (82d)$$

$$c_{1I} = \frac{1}{2p^3\mathcal{D}} [-\ell(\sin(p\ell - bp) \sin bp)(\sin(p\xi - bp) - 2p\xi + bp + \sin p\xi \cos bp) - \ell(\sin(p\ell - bp))(\sin bp - bp \cos bp)(\cos p\xi \sin bp - 2 \sin p\xi \cos bp) - b\ell(\sin(p\ell - bp) \sin bp)(p \cos p\xi \cos bp - p + 2p \sin p\xi \sin bp)], \quad (82e)$$

$$c_{2I} = \frac{1}{2p^3\mathcal{D}} [(\sin(p\ell - bp) \sin bp)(\sin(p\xi - bp) - 2p\xi + bp + \sin p\xi \cos bp) + (\sin(p\ell - bp))(\sin bp - bp \cos bp)(\cos p\xi \sin bp - 2 \sin p\xi \cos bp) + b(\sin(p\ell - bp) \sin bp)(p \cos p\xi \cos bp - p + 2p \sin p\xi \sin bp)], \quad (82f)$$

$$c_{3I} = \frac{1}{2p^3\mathcal{D}} [-(b - \ell)(\sin p\ell \sin bp)(\sin(p\xi - bp) - 2p\xi + bp + \sin p\xi \cos bp) - (b - \ell)(\sin p\ell)(\sin bp - bp \cos bp)(\cos p\xi \sin bp - 2 \sin p\xi \cos bp) - b(b - \ell)(\sin p\ell \sin bp)(p \cos p\xi \cos bp - p + 2p \sin p\xi \sin bp)], \quad (82g)$$

$$c_{4I} = \frac{1}{2p^3\mathcal{D}} [(b - \ell) \cos p\ell \sin bp(\sin(p\xi - bp) - 2p\xi + bp + \sin p\xi \cos bp) - (b - \ell) \cos p\ell(\sin bp - bp \cos bp)(\cos p\xi \sin bp - 2 \sin p\xi \cos bp) + b(b - \ell) \cos p\ell \sin bp(p \cos p\xi \cos bp - p + 2p \sin p\xi \sin bp)], \quad (82h)$$

where

$$\mathcal{D} = bp(\ell - b) \sin p\ell - \ell \sin bp \sin(p\ell - bp). \quad (83)$$

With a_{1I}, \dots, c_{4I} we can calculate the Green functions using the corresponding definition:

$$G_{1I}(x, \xi) = \sum_{\ell=1}^4 (a_{\ell I}(\xi) \pm b_{\ell I}(\xi)) w_{\ell}(x) \quad x \in (0, b) \quad (84a)$$

and

$$G_{2I}(x, \xi) = \sum_{\ell=1}^4 c_{\ell I}(\xi) w_{\ell}(x) \quad x \in (b, \ell). \quad (84b)$$

Calculation of the Green function if $\xi \in (b, \ell)$:

The equation system for $b_{\ell II}$ coincides with the equation system (79). Hence $b_{\ell II} = b_{\ell I}$ and the solutions are given by equation (80).

The boundary conditions (67a) and (67a) at $x = 0$ yield:

$$c_{1II} = c_{3II} = 0.$$

The continuity conditions at $x = b$ and the boundary conditions at $x = \ell$ lead to the following equation system – we have omitted the details of the calculation:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b & \sin pb \\ 1 & b & \cos pb & \sin pb & 0 & 0 \\ 0 & 1 & -p \sin pb & p \cos pb & -1 & -p \cos pb \\ 0 & 0 & -\cos pb & -\sin pb & 0 & \sin pb \\ 1 & \ell & \cos p\ell & \sin p\ell & 0 & 0 \\ 0 & 0 & -\cos p\ell & -\sin p\ell & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4I} \\ c_{2II} \\ c_{4II} \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} 0 \\ -p\xi + pb + \sin(p\xi - bp) \\ p - p \cos(p\xi - bp) \\ -\sin(p\xi - bp) \\ p\xi - p\ell - \sin(p\xi - p\ell) \\ \sin(p\xi - p\ell) \end{bmatrix}. \quad (85)$$

The solutions needed for calculating the Green function are gathered here:

$$\begin{aligned} a_{1II} = \frac{1}{2p^3 \mathcal{D}} & [(\ell \sin(p\ell - bp) \sin bp - b p \ell \sin p\ell) (\sin(p\xi - bp) - p\xi + bp) + \\ & + (\sin(p\xi - p\ell)) (b^2 p \sin p\ell - b p \ell \sin bp) - b^2 p (\sin p\ell) (\sin(p\xi - p\ell) - p\xi + p\ell) - \\ & - \ell (\sin(p\xi - bp) \sin(p\ell - bp)) (\sin bp - bp \cos bp) - \\ & - b \ell (\sin(p\ell - bp) \sin bp) (p - p \cos(p\xi - bp))], \quad (86a) \end{aligned}$$

$$\begin{aligned} a_{2II} = \frac{1}{2p^3 \mathcal{D}} & [(\sin(p\xi - p\ell)) (\sin(p\ell - bp) \sin bp - bp (\sin p\ell - \sin bp)) - \\ & - (\sin(p\ell - bp) \sin bp - bp \sin p\ell) (\sin(p\xi - bp) - p\xi + bp) - \\ & - (\sin(p\ell - bp) \sin bp - bp \sin p\ell) (\sin(p\xi - p\ell) - p\xi + p\ell) + \\ & + (\sin(p\xi - bp) \sin(p\ell - bp)) (\sin bp - bp \cos bp) + b (\sin(p\ell - bp) \sin bp) (p - p \cos(p\xi - bp))], \quad (86b) \end{aligned}$$

$$\begin{aligned} a_{3II} = \frac{1}{2p^3 \mathcal{D}} & [b (\sin p\ell \sin bp) (\sin(p\xi - bp) - p\xi + bp) - \\ & - (\sin(p\xi - p\ell) \sin bp) (b \sin p\ell - \ell \sin bp) + b (\sin p\ell \sin bp) (\sin(p\xi - p\ell) - p\xi + p\ell) - \\ & - (\sin(p\xi - bp) \sin p\ell) (\sin bp - bp \cos bp) (b - \ell) - \\ & - b (\sin p\ell \sin bp) (b - \ell) (p - p \cos(p\xi - bp))], \quad (86c) \end{aligned}$$

$$\begin{aligned} a_{4II} = \frac{1}{2p^3 \mathcal{D}} & [(\sin(p\xi - p\ell)) ((\sin bp) (b \cos p\ell - \ell \cos bp) - bp (b - \ell)) - \\ & - b (\cos p\ell \sin bp) (\sin(p\xi - bp) - p\xi + bp) - b (\cos p\ell \sin bp) (\sin(p\xi - p\ell) - p\xi + p\ell) + \\ & + (\sin(p\xi - bp) \cos p\ell) (\sin bp - bp \cos bp) (b - \ell) + \\ & + b (\cos p\ell \sin bp) (b - \ell) (p - p \cos(p\xi - bp))], \quad (86d) \end{aligned}$$

$$c_{1II} = 0, \quad (86e)$$

$$\begin{aligned} c_{2II} = \frac{1}{2p^3 \mathcal{D}} & [(\sin(p\xi - p\ell)) (\sin(p\ell - bp) \sin bp + p (\sin bp) (b - \ell)) + \\ & + (\sin(p\xi - bp)) (\sin(p\ell - bp) \sin bp + p (\cos(p\ell - bp) \sin bp) (b - \ell)) - \\ & - (\sin(p\ell - bp) \sin bp) (\sin(p\xi - p\ell) - p\xi + p\ell) - (\sin(p\ell - bp) \sin bp) (\sin(p\xi - bp) - p\xi + bp) + \\ & + (\sin(p\ell - bp) \sin bp) (b - \ell) (p - p \cos(p\xi - bp))], \quad (86f) \end{aligned}$$

$$c_{3II} = 0, \quad (86g)$$

$$\begin{aligned} c_{4II} = \frac{1}{2p^3 \mathcal{D}} & [b (\sin(p\ell - bp)) (\sin(p\xi - bp) - p\xi + bp) - \\ & - (\sin(p\xi - bp)) (b \sin(p\ell - bp) + bp (\cos(p\ell - bp)) (b - \ell)) \\ & - (\sin(p\xi - p\ell)) (b \sin(p\ell - bp) + bp (b - \ell)) + b (\sin(p\ell - bp)) (\sin(p\xi - p\ell) - p\xi + p\ell) - \\ & - b (\sin(p\ell - bp)) (b - \ell) (p - p \cos(p\xi - bp))], \quad (86h) \end{aligned}$$

where

$$\mathcal{D} = bp(b - \ell) \sin p\ell - \ell \sin bp \sin (p\ell - bp). \quad (87)$$

With a_{1II}, \dots, c_{4II} we can calculate the Green functions using the corresponding definition:

$$G_{2I}(x, \xi) = \sum_{\ell=1}^4 c_{\ell I}(\xi) w_{\ell}(x) \quad x \in (0, b) \quad (88a)$$

and

$$G_{1I}(x, \xi) = \sum_{\ell=1}^4 (a_{\ell II}(\xi) \pm b_{\ell II}(\xi)) w_{\ell}(x) \quad x \in (b, \ell). \quad (88b)$$

6.2.3. Problem 3. If the axial force is nonzero ($N \neq 0$) but a tensile one equilibrium problems of heterogeneous beams are governed by the differential equation

$$L(w) = \frac{d^4 w}{dx^4} - \mathcal{N} \frac{d^2 w}{dx^2} = \hat{f}. \quad (89)$$

For the beam shown in Figure 2 this equation is associated with the following boundary and continuity conditions:

$$w(0) = 0, \quad w^{(2)}(0) = 0; \quad w(\ell) = 0, \quad w^{(2)}(\ell) = 0, \quad (90a)$$

$$\begin{aligned} w(b-0) &= 0, & w(b+0) &= 0, \\ w^{(1)}(b-0) &= w^{(1)}(b+0), \\ w^{(2)}(b-0) &= w^{(2)}(b+0). \end{aligned} \quad (90b)$$

The three point boundary value problem (76), (77) is self-adjoint. Our aim is to find the Green function.

The linearly independent particular solutions of equation $L(w) = 0$ are given by

$$w_1 = 1, \quad w_2 = x, \quad w_3 = \cosh px, \quad w_4 = \sinh px, \quad p = \sqrt{\frac{N}{I_{ey}}}, \quad N > 0. \quad (91)$$

The general solution and its derivatives are as follows:

$$\begin{aligned} w &= a_1 + a_2 x + a_3 \cosh px + a_4 \sinh px, \\ \frac{dw}{dx} &= a_2 + pa_3 \sinh px + pa_4 \cosh px, \\ \frac{d^2 w}{dx^2} &= p^2 a_3 \cosh px + p^2 a_4 \sinh px, \\ \frac{d^3 w}{dx^3} &= p^3 a_3 \sinh px + p^3 a_4 \cosh px. \end{aligned} \quad (92)$$

Calculation of the Green function if $\xi \in (a = 0, b)$:

The continuity and discontinuity conditions (36) results in the following equation system for $b_{\ell I}$:

$$\begin{bmatrix} 1 & \xi & \cosh p\xi & \sinh p\xi \\ 0 & 1 & p \sinh p\xi & p \cosh p\xi \\ 0 & 0 & p^2 \cosh p\xi & p^2 \sinh p\xi \\ 0 & 0 & p^3 \sinh p\xi & p^3 \cosh p\xi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \quad (93)$$

from where

$$\begin{bmatrix} b_{1I} \\ b_{2I} \\ b_{3I} \\ b_{4I} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2p^2} \xi \\ \frac{1}{2p^2} \\ \frac{1}{2p^3} \sinh p\xi \\ -\frac{1}{2p^3} \cosh p\xi \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} -p\xi \\ p \\ \sinh p\xi \\ \cosh p\xi \end{bmatrix}. \quad (94)$$

Continuity conditions (54b) and boundary conditions (54a)₂ result in the following equation system:

$$\begin{bmatrix} b & \sinh pb & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b & \cosh pb & \sinh pb \\ 1 & p \cosh pb & 0 & -1 & -p \sinh pb & -p \cosh pb \\ 0 & \sinh pb & 0 & 0 & -\cosh pb & -\sinh pb \\ 0 & 0 & 1 & \ell & \cosh p\ell & \sinh p\ell \\ 0 & 0 & 0 & 0 & \cosh p\ell & \sinh p\ell \end{bmatrix} \begin{bmatrix} a_{2I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4i} \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} -2p\xi + pb + \sinh(p\xi - bp) + \sinh p\xi \cosh pb \\ 0 \\ p + 2p \sinh p\xi \sinh pb - p \cosh p\xi \cosh pb \\ 2 \sinh p\xi \cos pb - \cosh p\xi \sin pb \\ 0 \\ 0 \end{bmatrix}. \quad (95)$$

The solutions needed for calculating the Green function are given by the following equations:

$$a_{1I} = \frac{1}{2p^3} p\xi, \quad (96a)$$

$$\begin{aligned} a_{2I} = \frac{1}{2p^3 \mathcal{D}} & [(\sinh bp \sinh(p\ell - bp) + p(\sinh bp \cosh(p\ell - bp))(b - \ell))(\cosh p\xi \sin bp - 2 \sinh p\xi \cos bp) + \\ & + (\sinh bp \sinh(p\ell - bp) + p(\sinh p\ell)(b - \ell))(\sinh(p\xi - bp) - 2p\xi + \cosh bp \sinh p\xi + bp) - \\ & - (\sinh bp \sinh(p\ell - bp))(b - \ell)(p - p \cosh bp \cosh p\xi + 2p \sinh bp \sinh p\xi)], \end{aligned} \quad (96b)$$

$$a_{3I} = -\frac{1}{2p^3} \sinh p\xi, \quad (96c)$$

$$\begin{aligned} a_{4I} = \frac{1}{2p^3 \mathcal{D}} & [b(\sinh(p\ell + bp))(b - \ell)(p - p \cosh bp \cosh p\xi + 2p \sinh bp \sinh p\xi) - \\ & - (\sinh(p\ell - bp))(b - \ell)(\sinh(p\xi - bp) - 2p\xi + \cosh bp \sinh p\xi + bp) - \\ & - (\cosh p\xi \sin bp - 2 \sinh p\xi \cos bp)(b \sinh(p\ell - bp) + bp(\cosh(p\ell - bp))(b - \ell))], \end{aligned} \quad (96d)$$

$$\begin{aligned} c_{1I} = \frac{1}{2p^3 \mathcal{D}} & [b\ell(\sinh bp \sinh(p\ell - bp))(p - p \cosh bp \cosh p\xi + 2p \sinh bp \sinh p\xi) - \\ & - \ell(\sinh(p\ell - bp))(\sinh bp - bp \cosh bp)(\cosh p\xi \sin bp - 2 \sinh p\xi \cos bp) - \\ & - \ell(\sinh bp \sinh(p\ell - bp))(\sinh(p\xi - bp) - 2p\xi + \cosh bp \sinh p\xi + bp)], \end{aligned} \quad (96e)$$

$$\begin{aligned} c_{2I} = \frac{1}{2p^3 \mathcal{D}} & [(\sinh bp \sinh(p\ell - bp))(\sinh(p\xi - bp) - 2p\xi + \cosh bp \sinh p\xi + bp) + \\ & + (\sinh(p\ell - bp))(\sinh bp - bp \cosh bp)(\cosh p\xi \sin bp - 2 \sinh p\xi \cos bp) - \\ & - b(\sinh bp \sinh(p\ell - bp))(p - p \cosh bp \cosh p\xi + 2p \sinh bp \sinh p\xi)], \end{aligned} \quad (96f)$$

$$\begin{aligned} c_{3I} = \frac{1}{2p^3 \mathcal{D}} & [b(\sinh bp \sinh p\ell)(b - \ell)(p - p \cosh bp \cosh p\xi + 2p \sinh bp \sinh p\xi) - \\ & - (\sinh p\ell)(\sinh bp - bp \cosh bp)(b - \ell)(\cosh p\xi \sin bp - 2 \sinh p\xi \cos bp) - \\ & - (\sinh bp \sinh p\ell)(b - \ell)(\sinh(p\xi - bp) - 2p\xi + \cosh bp \sinh p\xi + bp)], \end{aligned} \quad (96g)$$

$$\begin{aligned} c_{4I} = \frac{1}{2p^3 \mathcal{D}} & [(\sinh bp \cosh p\ell)(b - \ell)(\sinh(p\xi - bp) - 2p\xi + \cosh bp \sinh p\xi + bp) + \\ & + (\cosh p\ell)(\sinh bp - bp \cosh bp)(b - \ell)(\cosh p\xi \sin bp - 2 \sinh p\xi \cos bp) - \\ & - b(\sinh bp \cosh p\ell)(b - \ell)(p - p \cosh bp \cosh p\xi + 2p \sinh bp \sinh p\xi)], \end{aligned} \quad (96h)$$

where

$$\mathcal{D} = bp(b - \ell) \sinh p\ell + \ell \sinh bp \sinh(p\ell - bp). \quad (97)$$

With a_{1I}, \dots, c_{4I} and \mathcal{D} equations (84a) and (84b) can be used for determining the Green function if $\xi \in [0, a]$.

Calculation of the Green function if $\xi \in (b, c)$:

The equation system for $b_{\ell II}$ coincides with the equation system (93). Hence $b_{\ell II} = b_{\ell I}$ and the solutions are given by equation (94).

The boundary conditions (67a) and (67a) at $x = 0$ yield:

$$c_{1II} = c_{3II} = 0.$$

The continuity conditions at $x = b$ and the boundary conditions at $x = \ell$ lead to the following equation system – we have omitted the details of the calculation:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b & \sinh pb \\ 1 & b & \cosh pb & \sinh pb & 0 & 0 \\ 0 & 1 & p \sinh pb & p \cosh pb & -1 & -p \cosh pb \\ 0 & 0 & \cosh pb & \sinh pb & 0 & -\sinh pb \\ 1 & \ell & \cosh p\ell & \sinh p\ell & 0 & 0 \\ 0 & 0 & \cosh p\ell & \sinh p\ell & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4I} \\ c_{2II} \\ c_{4II} \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} 0 \\ \xi - pb - \sinh(p\xi - bp) \\ -p + p \cosh(p\xi - bp) \\ -\sinh(p\xi - bp) \\ -p\xi + p\ell + \sinh(p\xi - p\ell) \\ \sinh(p\xi - p\ell) \end{bmatrix}. \quad (98)$$

The solutions needed for calculating the Green function are as follows:

$$\begin{aligned} a_{1II} = \frac{1}{2p^3\mathcal{D}} & [b^2p(\sinh p\ell)(\sinh(p\xi - p\ell) - p\xi + p\ell) - (\sinh(p\xi - p\ell))(b^2p \sinh p\ell - bp\ell \sinh bp) - \\ & - (\ell \sinh bp \sinh(p\ell - bp) - bp\ell \sinh p\ell)(\sinh(p\xi - bp) - \xi + bp) + \\ & + \ell(\sinh(p\xi - bp) \sinh(p\ell - bp))(\sinh bp - bp \cosh bp) + \\ & + b(\ell \sinh bp \sinh(p\ell - bp))(p - p \cosh(p\xi - bp))], \quad (99a) \end{aligned}$$

$$\begin{aligned} a_{2II} = \frac{1}{2p^3\mathcal{D}} & [(\sinh bp \sinh(p\ell - bp) - bp \sinh p\ell)(\sinh(p\xi - bp) - \xi + bp) - \\ & - (\sinh(p\xi - p\ell))(\sinh bp \sinh(p\ell - bp) + bp(\sinh bp - \sinh p\ell)) + \\ & + (\sinh bp \sinh(p\ell - bp) - bp \sinh p\ell)(\sinh(p\xi - p\ell) - p\xi + p\ell) - \\ & - (\sinh(p\xi - bp) \sinh(p\ell - bp))(\sinh bp - bp \cosh bp) - b(\sinh bp \sinh(p\ell - bp))(p - p \cosh(p\xi - bp))], \quad (99b) \end{aligned}$$

$$\begin{aligned} a_{3II} = \frac{1}{2p^3\mathcal{D}} & [(\sinh bp \sinh(p\xi - p\ell))(b \sinh p\ell - \ell \sinh bp) - b(\sinh bp \sinh p\ell)(\sinh(p\xi - bp) - \xi + bp) + \\ & + (\sinh(p\xi - bp) \sinh p\ell)(\sinh bp - bp \cosh bp)(b - \ell) - \\ & b(\sinh bp \sinh p\ell)(\sinh(p\xi - p\ell) - p\xi + p\ell) + \\ & + b(\sinh bp \sinh p\ell)(p - p \cosh(p\xi - bp))(b - \ell)], \quad (99c) \end{aligned}$$

$$\begin{aligned} a_{4II} = \frac{1}{2p^3\mathcal{D}} & [b(\sinh bp \cosh p\ell)(\sinh(p\xi - bp) - \xi + bp) - \\ -(\sinh(p\xi - p\ell)) & ((\sinh bp)(b \cosh p\ell - \ell \cosh bp) - bp(b - \ell)) - (\sinh(p\xi - bp) \cosh p\ell)(\sinh bp - bp \cosh bp)(b - \ell) + \\ & + b(\sinh bp \cosh p\ell)(\sinh(p\xi - p\ell) - p\xi + p\ell) - \\ & - b(\sinh bp \cosh p\ell)(p - p \cosh(p\xi - bp))(b - \ell)], \quad (99d) \end{aligned}$$

$$c_{1II} = 0, \quad (99e)$$

$$\begin{aligned} c_{2II} = \frac{1}{2p^3\mathcal{D}} & [(\sinh bp \sinh(p\ell - bp))(\sinh(p\xi - p\ell) - p\xi + p\ell) - \\ & - (\sinh(p\xi - bp))(\sinh bp \sinh(p\ell - bp) + p(\sinh bp \cosh(p\ell - bp))(b - \ell)) - \\ -(\sinh bp \sinh(p\ell - bp)) & (\sinh(p\xi - bp) - \xi + bp) - (\sinh(p\xi - p\ell))(\sinh bp \sinh(p\ell - bp) + p(\sinh bp)(b - \ell)) - \\ & - (\sinh bp \sinh(p\ell - bp))(p - p \cosh(p\xi - bp))(b - \ell)], \quad (99f) \end{aligned}$$

$$c_{3II} = 0, \quad (99g)$$

$$\begin{aligned} c_{4II} = \frac{1}{2p^3\mathcal{D}} & [(\sinh(p\xi - p\ell))(b \sinh(p\ell - bp) - bp(b - \ell)) + \\ & + (\sinh(p\xi - bp))(b \sinh(p\ell - bp) + bp \cosh(p\ell - bp)(b - \ell)) - b(\sinh(p\ell - bp))(\sinh(p\xi - bp) - \xi + bp) - \\ & - b(\sinh(p\ell - bp))(\sinh(p\xi - p\ell) - p\xi + p\ell) + b(\sinh(p\ell - bp))(p - p \cosh(p\xi - bp))(b - \ell)]. \quad (99h) \end{aligned}$$

7. EIGENVALUE PROBLEMS WITH GREEN FUNCTIONS

7.1. Transformation to a Fredholm integral equation. Consider the differential equation

$$K[y(x)] = \lambda g_0(x)y(x) \quad (100)$$

where $y(x)$ is the unknown function, λ is a parameter (the eigenvalue sought). The differential operator $K[y(x)]$ is defined by the relationship:

$$K[y(x)] = \sum_{n=0}^{\kappa} (-1)^n [f_n(x)y^{(n)}(x)]^{(n)}, \quad \frac{d^n(\dots)}{dx^n} = (\dots)^{(n)} \quad (101)$$

in which the real function $f_\nu(x)$ is differentiable continuously κ times ($\kappa \geq 1$), while the real function $g_0(x)$ continuous and

$$f_\kappa(x) \neq 0, \quad g_0(x) > 0 \quad \text{if } x \in [a, c]. \quad (102)$$

Differential equation (100) is associated with homogenous boundary conditions (2a), and continuity conditions (2b). Equations (100) and (2) determine a three point eigenvalue problem. It is assumed that the operator $K[y(x)]$ is self-adjoint under the boundary and continuity conditions mentioned above. Hence, the Green function that belongs to the operator $K[y(x)]$ is symmetric: $G(x, \xi) = G(\xi, x)$.

With the Green function it follows from (22) that the solution of the three point eigenvalue problem (100), (2) should satisfy the equation

$$\sqrt{g_0(x)} y(x) = \lambda \int_a^c \sqrt{g_0(x)} G(x, \xi) \sqrt{g_0(\xi)} y(\xi) d\xi. \quad (103)$$

By introducing the new functions

$$\mathcal{K}(x, \xi) = \sqrt{g_0(x)} G(x, \xi) \sqrt{g_0(\xi)} = \mathcal{K}(\xi, x), \quad \mathcal{Y} = \sqrt{g_0(x)} y(x) \quad (104)$$

we can rewrite equation (103) into the following form:

$$\mathcal{Y}(x) = \lambda \int_a^c \mathcal{K}(x, \xi) \mathcal{Y}(\xi) d\xi. \quad (105)$$

Equation (105) is a homogenous Fredholm integral equation with a symmetric kernel. In addition to this it is equivalent to the original three point boundary value problem.

7.2. Algorithm for the numerical solution. The present section investigates the problem of how to find a numerical solution for the eigenvalue problem determined by the Fredholm integral equation (105) by utilizing the boundary element technique which is a subinterval method [17] similarly to the finite element method. This means that the interval $[a, c]$, in which the solution for $y(x)$ in the equation

$$\chi \mathcal{Y}(x) = \int_a^c \mathcal{K}(x, \xi) \mathcal{Y}(\xi) dx, \quad \chi = \frac{1}{\lambda}, \quad (106)$$

is sought, is divided into parts (subintervals) called elements. Figure 4 depicts the interval $[a, c]$ which is now divided into $n_e = 5$ elements. These elements and their lengths are denoted in the same way by $\mathfrak{L}_1, \dots, \mathfrak{L}_{n_e}$. The points where the unknown function $\mathcal{Y}(x)$ is considered are the nodes (or nodal points) taken at the extremes (or ends) and the midpoint of an element. Elements with one node at the midpoint, or with two nodes at the end points can also be applied. Since more nodes results in greater accuracy and slightly less elements the three node elements are preferred in the sequel.

The nodal points are numbered. Locally by 1, 2 and 3, globally by $i = 1, 2, \dots, 2n_e + 1$. Figure 4 shows both the local and the global node numbers.

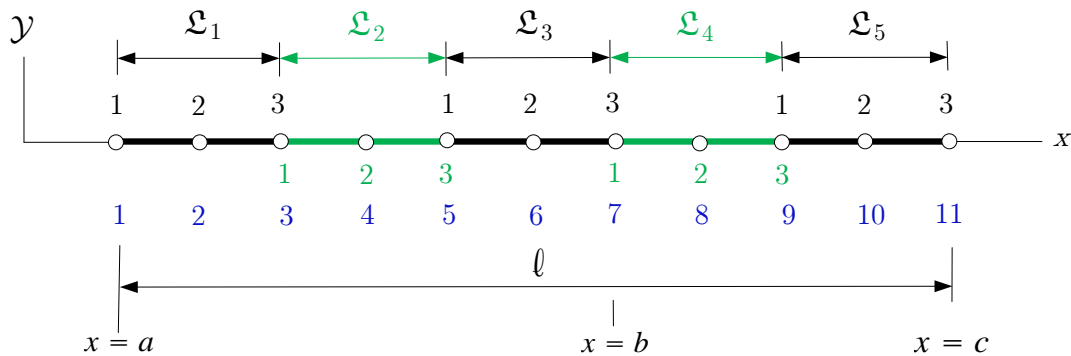


FIGURE 4. Subintervals for the numerical solution

The ξ coordinates of the local nodes and the values of the unknown function $y(x)$ at these nodes are the elements of the column matrices ξ^e and \mathbf{y}^e :

$$\xi^e = \begin{bmatrix} \xi_1^e \\ \xi_2^e \\ \xi_3^e \end{bmatrix}, \quad \mathbf{y}^e = \begin{bmatrix} \mathcal{Y}_1^e \\ \mathcal{Y}_2^e \\ \mathcal{Y}_3^e \end{bmatrix}, \quad (e = 1, 2, \dots, n_e). \quad (107)$$

The interpolation functions N_1 , N_2 and N_3 are defined by the following equations:

$$N_1(\eta) = \frac{1}{2}\eta(\eta - 1), \quad N_2(\eta) = 1 - \eta^2, \quad N_3(\eta) = \frac{1}{2}\eta(\eta + 1), \quad \eta \in [-1, 1]. \quad (108)$$

Node number	η	N_1	N_2	N_3
1	-1	1	0	0
2	0	0	1	0
3	1	0	0	1

TABLE 1. The typical interpolation function values

Table 1 contains the values of the interpolation functions for $\eta = -1$, $\eta = 0$ and $\eta = 1$. Hence the unknown function $\mathcal{Y}(x)$ can be approximated quadratically on the element \mathcal{L}_e by the equation

$$\mathcal{Y}^e(\eta) = [N_1(\eta) \mid N_2(\eta) \mid N_3(\eta)] \begin{bmatrix} \mathcal{Y}_1^e \\ \mathcal{Y}_2^e \\ \mathcal{Y}_3^e \end{bmatrix} = \mathbf{N}_{(1 \times 3)} \mathbf{y}_{(3 \times 1)}^e \quad (109a)$$

where

$$\mathbf{N}_{(1 \times 3)} = [N_1(\eta) \mid N_2(\eta) \mid N_3(\eta)]. \quad (109b)$$

As regards the geometry we can write in the same manner

$$\begin{aligned} \xi^e(\eta) &= [N_1(\eta) \mid N_2(\eta) \mid N_3(\eta)] \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \end{bmatrix} = \mathbf{N}_{(1 \times 3)} \mathbf{x}_{(3 \times 1)}^e = \\ &= \frac{1}{2}\eta^2 (x_1^e - 2x_2^e + x_3^e) + \frac{1}{2}\eta (x_3^e - x_1^e) + x_2^e = \underset{x_2^e = \frac{1}{2}(x_1^e + x_3^e)}{\uparrow} = \frac{1}{2}\eta (x_3^e - x_1^e) + x_2^e. \end{aligned} \quad (110)$$

Thus

$$\begin{aligned} d\xi^e &= \left(\frac{dN_1}{d\eta} x_1^e + \frac{dN_2}{d\eta} x_2^e + \frac{dN_3}{d\eta} x_3^e \right) d\eta = \frac{1}{2} (x_3^e - x_1^e) d\eta = J d\eta, \\ J &= \frac{1}{2} (x_3^e - x_1^e) = \frac{\mathcal{L}_e}{2}. \end{aligned} \quad (111)$$

Integrating element by element in (106) and substituting then (109), (111) we get

$$\begin{aligned} \chi \mathcal{Y}(x) &= \sum_{e=1}^{n_e} \int_{\mathcal{L}_e} \mathcal{K} [x, \xi(\eta)] [N_1(\eta) | N_2(\eta) | N_3(\eta)] \mathcal{J}(\eta) d\eta \begin{bmatrix} \mathcal{Y}_1^e \\ \mathcal{Y}_2^e \\ \mathcal{Y}_3^e \end{bmatrix} = \\ &= \sum_{e=1}^{n_{be}} \int_{\eta=-1}^{\eta=1} \mathcal{K} [x, \xi(\eta)] \mathbf{N}_{(1 \times 3)}(\eta) \mathcal{J}(\eta) d\eta \mathbf{y}_{(3 \times 1)}^e. \end{aligned} \quad (112)$$

Let us denote the coordinates of the nodes by x_i ($i = 1, 2, \dots, 2n_e + 1$). Then it follows from equation (112) that

$$\chi \mathcal{Y}(x_i) = \sum_{e=1}^{n_e} \int_{\eta=-1}^{\eta=1} \mathcal{K} [x_i, \xi(\eta)] [N_1(\eta) | N_2(\eta) | N_3(\eta)] \mathcal{J}(\eta) d\eta \begin{bmatrix} \mathcal{Y}_1^e \\ \mathcal{Y}_2^e \\ \mathcal{Y}_3^e \end{bmatrix}, \quad (i = 1, 2, \dots, 2n_e + 1). \quad (113)$$

For our later consideration we shall introduce the following notations

$$k_r^{ie} = \int_{\eta=-1}^{\eta=1} \mathcal{K} [x_i, \xi(\eta)] N_r(\eta) \mathcal{J}(\eta) d\eta, \quad \mathcal{Y}(x_i) = y_i, \quad (i = 1, 2, 3, \dots, 2n_e + 1; r = 1, 2, 3) \quad (114)$$

where i identifies the global number of the node, e is the number of the element over which the integral is taken and r is the number of the interpolation function. Using these notations equation (113) can be rewritten into the following form:

$$\left[k_1^{i1} \mid k_2^{i1} \mid k_3^{i1} + k_1^{i2} \mid k_2^{i2} \mid k_3^{i2} + k_1^{i3} \mid k_1^{i3} \mid \dots \mid k_3^{i, n_e - 1} + k_1^{i n_e} \mid k_2^{i n_e} \mid k_3^{i n_e} \right] \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{2n_e} \\ y_{2n_e + 1} \end{bmatrix} - \chi y_i = 0, \quad (i = 1, 2, 3, \dots, 2n_e + 1). \quad (115)$$

Here we have as many equations as there are unknowns.

For the sake of making the structure of these equations more clear it is worth introducing further notations:

$$k_{i1} = k_1^{i1}, \quad k_{i2} = k_2^{i1}, \quad k_{i3} = k_3^{i1} + k_1^{i2}, \quad k_{i4} = k_2^{i2}, \quad k_{i5} = k_3^{i2} + k_1^{i3}, \dots, \quad k_{i, 2n_e} = k_2^{i n_e}, \quad k_{i, 2n_e + 1} = k_3^{i n_e}. \quad (116)$$

If we use these notations and unite equations (115) we arrive at the following matrix equation:

$$\left\{ \left[\begin{array}{cccc} k_{11} & k_{12} & \dots & k_{1, 2n_e + 1} \\ k_{21} & k_{22} & \dots & k_{2, 2n_e + 1} \\ \dots & \dots & \dots & \dots \\ k_{n_e 1} & k_{n_e 2} & \dots & k_{n_e, 2n_e + 1} \end{array} \right] - \chi \left[\begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{array} \right] \right\} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_{2n_e + 1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}. \quad (117)$$

Equation (117) is an algebraic eigenvalue problem with $\chi = 1/\lambda$ as eigenvalue. After solving it numerically we have both the eigenvalues $\chi_r = 1/\lambda_r$ and the corresponding eigenvectors

$$\mathbf{y}_r^T = [y_{1r} \mid y_{2r} \mid y_{3r} \mid \dots \mid y_{2n_e + 1, r}], \quad (r = 1, 2, 3, \dots, 2n_e + 1).$$

The last issue we have to deal with is that of the numerical integration. Introduce the notation

$$\phi(\eta) = \mathcal{K} [x_i, \xi(\eta)] N_r(\eta) \mathcal{J}(\eta). \quad (118)$$

With (118) we can rewrite integral (114) into the following form:

$$k_r^{ie} = \int_{\eta=-1}^{\eta=1} \mathcal{K} [x_i, \xi(\eta)] N_r(\eta) \mathcal{J}(\eta) d\eta = \int_{\eta=-1}^{\eta=1} \phi(\eta) d\eta \quad (119)$$

which can be computed by applying a sufficiently accurate Gaussian quadrature rule.

8. CONCLUDING REMARKS

A definition is provided for the Green functions of three point boundary value problems. The definition is a constructive one: on the basis of the definition the paper details the procedure that can be used to calculate them. The applicability of the procedure is demonstrated via three examples. In addition to this the existence of the Green function is also proven. The most important properties of the Green functions are presented too. Making use of the Green functions a class of the three point eigenvalue problems can be reduced to eigenvalue problems governed by homogenous Fredholm integral equations. A solution algorithm is suggested which results in an algebraic eigenvalue problem.

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