A new class of optimization problems related to contact interaction

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Abstract.

For some structures under service loads there is a need of precise control of local boundary displacement and/or its tangential gradient by an additional loading of one or two punches. Such problems exist in design of robot grippers or mechanical tools used in element assembling or in other mechanical processes. The punch interaction is assumed to be executed by a discrete set of pins or by a continuously distributed contact pressure. The optimal contact force and pressure distribution are defined in terms of assumed control function, for which contact shape is specified for both discrete and continuous punch action. For beam or plate structures three classes of control are considered. First, requiring by punch action the fixed load F_Q and displacement u_Q^* at a specified position, second, requiring the load-displacement evolution $F_Q = F_Q(u_Q^*)$ by the varying punch load and third, provide deflection and slope control at a specified position by a coordinated action of two punches. The reciprocal motion of a transverse pin attached to the beam is induced by varying punch forces. The punch position is specified by satisfying constraints on maximum punch pressure and equivalent Mises stress on the contact interface. Several illustrative examples are presented to illustrate punch control for different boundary supports and three control classes.

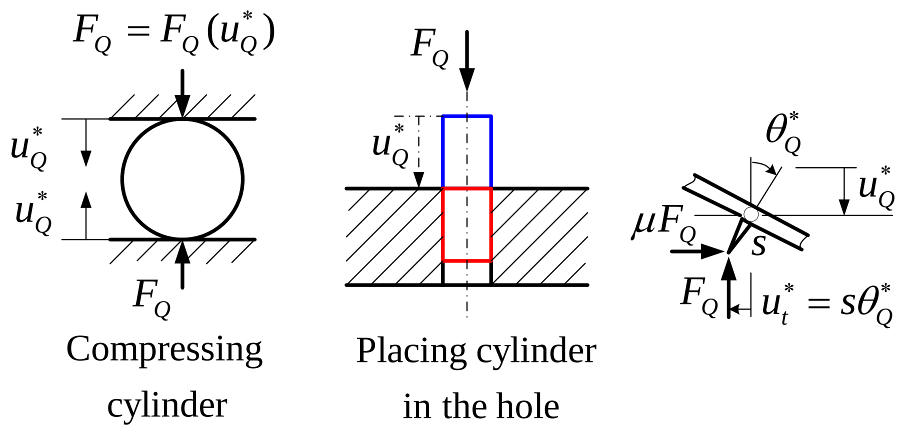
Key words. Contact problem, displacement and slope control, optimal pressure distribution, optimal contact shape.

1. Introduction

The design parameters in structural optimization are usually defined as material moduli, structure size, shape and topology characteristic parameters, supports, loads, inner links, reinforcement, cf. Banichuk [1]. The mathematical programming technique has been used by many authors for shape optimization of structures. Referring to contact problems, usually the peak contact pressure and the interface stress concentration have been minimized by using special mathematical programming techniques and contact shape sensitivity analysis [2]. In [3], and [4, 5] several classes of contact optimization problems have been considered with account for wear process.

A class of contact optimization problems for kinematical constraints has been treated in the paper by Páczelt [6]. Several classes of optimization problems have also been considered in the paper [5]: for axisymmetric punch shapes of arbitrary meridian profile the contact shape optimization problems were treated for specified punch displacement, prescribed punch load and for steady wear state conditions. In some examples the effective Mises stress was required to be below a prescribed ultimate stress. The numerical solutions have been obtained by applying a special iteration process using also the concept of partially controlled contact pressure [7]. The monographs of Goryacheva [3] and Wriggers [8] provide solid foundation for analytical and numerical methods of solution of contact problems,
including wear analysis. The finite element analysis is most frequently applied in solving contact problems, cf. Szabó and Babuska [9].

In this work a new class of optimization problems is formulated. It is required that at the same boundary point (or points) of a structure, the displacement and force are prescribed. To achieve this condition, the punch contact pressure action is applied at some location on the structure boundary. Then, the punch force and its location should be specified, combined with specified contact pressure distribution and required contact shape, satisfying the constraint set on the maximal contact pressure and the stress level at contacting material interfaces. This problem can exist in the design of robot elements, such as clippers and gardening or plantation tools for mechanical processing. In other words, the problem is reduced to a local displacement control in a structure subjected to service loads, such as an assembling robot gripper [10]. It is assumed that materials of the contacting bodies are linearly elastic, displacements and strains are small. The supporting constraints set on a structural element are most important in specifying a proper controlling punch action.



a) b) c)

Fig. 1. Typical robot gripper operations: a) compressing cylinder by required normal forces b) placing cylinder into a structural element hole. In both cases the load -displacement relation $F_Q = F_Q(u_Q^*)$ results from contact interaction. c) control of normal and tangential tool displacements

In mechanical engineering practice fairly abundant operation called "pick and place" is related to lifting the cylinder and placing in a new position, Fig. 1a. In this case, the cylinder is compressed along its diameter by two plates inducing normal contact displacements u_Q^* nonlinearly related to contact forces F_Q. In the assembling process of mechanical elements a typical operation is to place one body (cylinder) into a hole of another body, Fig.1b. In this case the cylinder must execute translation u_Q^* under increasing axial force F_Q induced by friction between cylinder and hole. In both operations the prescribed force and displacement values are required at the same point. In Fig. 1c the control of normal and tangential tool displacement is presented.

In the paper three classes of displacement control problems will be discussed. First, it is assumed that punch should apply the fixed loading assuring the required values of F_Q and u_Q^* at a specified position *Q*, where the interaction of the gripper with an object occurs. The optimal punch action inducing such control is discussed in Section 2. The second class of control requires the varying punch load in order to follow the relation $F_Q = F_Q(u_Q^*)$ resulting from gripper interaction with a structural element. Such varying load control is considered in Section 3 by assuming the action of two punches. In Section 4, the most advanced control by two punches applying fixed or varying loads is considered by requiring the control of both: normal displacement and its slope. Then a working tool attached to the beam can execute not only normal displacement, but also tangential displacement required by an executed technological process.

2. **Control of beam deflection at the loading point *Q***

Consider a beam shown in Fig. 2 with two support conditions at the left end A and with free right end B. In the first case the cantilever beam is built-in at its end A (it is referred to as *Beam I*). In the second case, the beam is allowed to slide vertically at its support A with constrained rotation (it is referred to as *Beam II*). In both cases the beam is loaded at point Q by the force F_Q, inducing the deflection $u_n^{(2)}$ in the $-z$ direction. To keep the deflection of Q at the required value u_Q^*, the discrete or continuous punch action is applied within the specified contact zone region Ω defined as an interval $L_1 \le x \le L_4$ of the length $l_c = L_4 - L_1$ The punch is allowed to translate in the normal direction $\boldsymbol{n}$ to the beam and exert contact pressure.

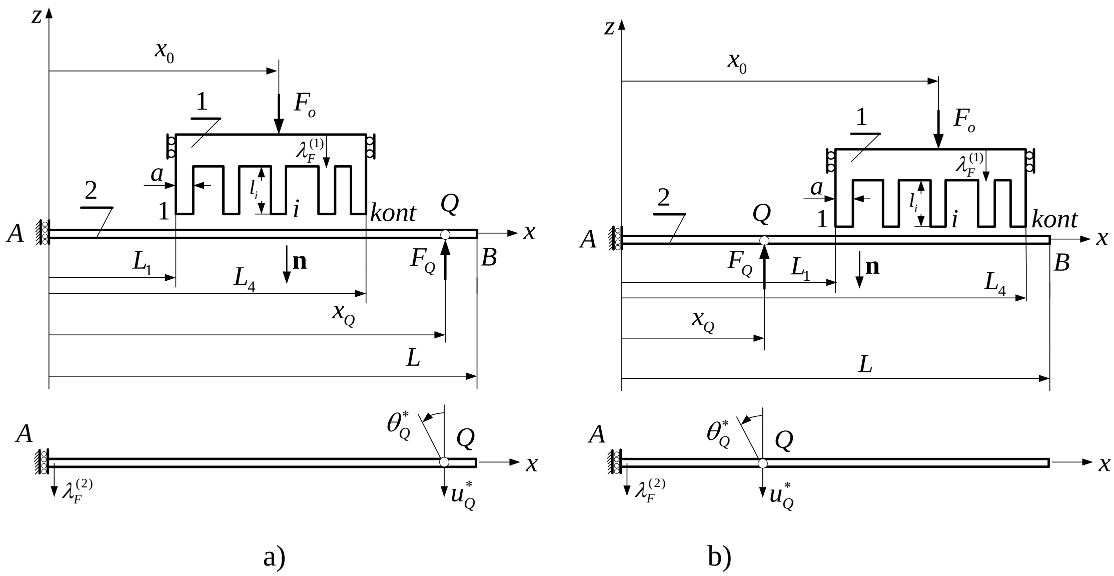


Fig. 2. Beam and punch system for transmission of load F_0 interacting with the force F_Q to induce vertical displacement u_Q^* and slope θ_Q^*. Beam 2 is allowed for rigid body displacement $\lambda_F^{(2)}$ and elastic deformation.

The beam cross section area is $A_b = a_b h_b$, inertia moment equals $I = a_b h_b^3/12$, and Young modulus is denoted by E .Two schemes correspond to a) $x_0 \le x_Q$ and b) $x_Q \le x_0$.

In our analysis the strength condition will be applied in specifying beam heights. An alternative design of varying heights is also considered. The contact pressure distribution is assumed in the form

$$p_n = c(x)\, p_{\max}, \quad c(x) \le 1, \quad \text{for} \quad x \in \Omega \qquad (1)$$

where $c(x)$ is the control function and p_{max} is the maximal pressure. Usually the control function is assumed and the constraint is set on the maximal pressure which depends on punch position and the contact zone length. Determination of the initial gap between punch and beam corresponding to pressure distribution (1) constitutes the objective of numerical analysis. For the first class of control problem two designs are considered. First, the punch center location is assumed as fixed and second, specifying punch position with account for the beam stress constraint $\sigma_{\max} \le \sigma_u$, where σ_u is the ultimate stress value.

The problem is solved in two main steps. First, using the stamp equilibrium condition at given control function $c(x)$ and constraint of normal displacement at the point Q. In this way $p_{\max}$ is specified, see equation (3). Second, from contact condition between the stamp and beam the initial gap between the contacting bodies is determined.

Consider first the case of punch action executed by a set of punch pins, as shown in Fig. 2, with thickness *a*, width *b* and cross section area *A* = *ab*. The forces between punch and beam are specified by Eq. (1), thus

$$P_j = Ac(x_j)\, p_{\max}\ , j = 1,\ 2, \ldots, kont \tag{2}$$

For *Beam II* under applied load $F_0 = F_Q$ on the punch, the maximal pressure results from the equilibrium condition

$$p_{\max} = \frac{F_0}{\sum_{j=1}^{kont} c(x_j)\, A} \tag{3}$$

but for *Beam I*, using equation $u_Q^* = \sum_{j=1}^{kont} H^{(2)}(x_Q, x_j) P_J + u_{n,load}^{(2)}$ we find

$$p_{\max} = \frac{u_Q^* - u_{n,load}^{(2)}}{\sum_{j=1}^{kont} H^{(2)}(x_Q, x_j) c(x_j)\, A} \tag{4}$$

where $H^{(2)}(x,s)$ is the influence Green function (see Appendix A), $u_{n,load}^{(2)} < 0$ is the displacement at point *Q* induced by the load F_Q. In the equation for u_Q^* the first term provides the deflection due to load F_0, and the second term provides the deflection due to load F_Q.

The *Beam II* displacement in the normal direction ***n*** along the axis –*z* equals

$$u_Q^* = \lambda_F^{(2)} + \sum_{j=1}^{kont} H^{(2)}(x_Q, x_j) P_J + u_{n,load}^{(2)} \tag{5}$$

because the beam end *A* executes the sliding displacement $\lambda_F^{(2)}$.

Using the influence function for calculation of $u_{n,load}^{(2)}$, we can write

$$u_{n,load}^{(2)} = -H^{(2)}(x_Q, x_Q) F_Q = -H_{Q,Q}^{(2)} F_Q \tag{6}$$

and the support displacement can be determined from (4)

$$\lambda_F^{(2)} = u_Q^* - \left(\sum_{j=1}^{kont} H^{(2)}(x_Q, x_j) \frac{c(x_j)}{\sum_{k=1}^{kont} c(x_k)} - H_{Q,Q}^{(2)} \right) F_Q \tag{7}$$

The contact condition (gap after deformation) between the punch and beam is

$$d_i = u_{in}^{(2)} - u_{in}^{(1)} + g_i^{(0)} = 0,\ i=1,\dots,kont \tag{8}$$

Using the influence Green functions, Eq. (7) can be expressed as follows

$$\lambda_F^{(2)} + \left(\sum_{j=1}^{kont} H(x_i, x_j) - H_{i,Q}^{(2)} \right) F_Q - \lambda_F^{(1)} + g_i^{(0)} = 0 \tag{9}$$

where $H(x_i, x_j) = \left(H^{(1)}(x_i, x_j) + H^{(2)}(x_i, x_j) \right) \dfrac{c(x_j)}{\sum_{k=1}^{kont} c(x_k)}$ (10)

Discretizing (7)-(9) we can write

$$\mathbf{d} = ({}^{iter}\mathbf{He} - \mathbf{h}_Q^{(2)}) F_Q + \mathbf{e}\lambda_F^{(2)} - \mathbf{e}\,{}^{(iter)}\lambda_F^{(1)} + {}^{(iter)}\mathbf{g}^{(0)} = \mathbf{0} \tag{11}$$

where $\mathbf{h}_Q^{(2),T} = \left[H_{1,Q}^{(2)}, H_{2,Q}^{(2)}, \dots, H_{i,Q}^{(2)}, \dots, H_{kont,Q}^{(2)} \right]$, $\mathbf{e}^T = [1, \dots, 1, \dots 1]$

Because F_Q and $\lambda_F^{(2)}$ are known (see (7)), then ${}^{iter}\mathbf{u}$ can be calculated, namely

$${}^{iter}\mathbf{u} = ({}^{iter}\mathbf{He} - \mathbf{h}_Q^{(2)}) F_Q + \mathbf{e}\lambda_F^{(2)} \tag{12}$$

and from the following equation

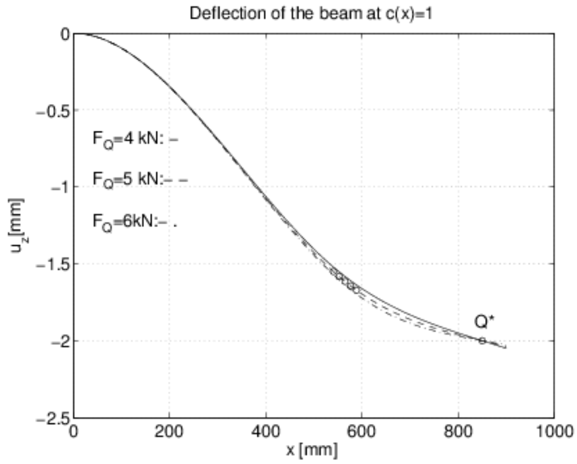
$${}^{iter}\mathbf{u} - \mathbf{e}\,{}^{(iter)}\lambda_F^{(1)} + {}^{(iter)}\mathbf{g}^{(0)} = \mathbf{0} \tag{13}$$

one can easily find ${}^{iter}\lambda_F^{(1)}$. When specifying ${}^{(iter)}\mathbf{g}^{(0)}$, suppose ${}^{(iter)}g_1{}^{(0)} = 0$, then ${}^{(iter)}\lambda_F^{(1)} = {}^{(iter)}u_1$ and from (12) we determine the initial gap ${}^{(iter)}\mathbf{g}^{(0)}$.

2.1 *Example: beam I for the discrete punch action*

Geometric parameters: $a_b = 20, h_b = 70\ mm$, $L = 950\ mm, x_Q = 850\ mm$, $L_1 = 220\ mm, L_4 = 280\ mm$. Punch pins cross section $A = a \cdot b = 5 \cdot 20 = 100\ mm^2$, $l_i = 50\ mm, i = 1,..,kont$, $kont$=5. Material parameters: Young modulus $E = 2 \cdot 10^5\ MPa$, $\sigma_u = 150\ MPa$.The punch centre position $x_0 = 250\ mm$ corresponds to the position parameter $\xi = x_0 / x_Q = 0.556$. The specified vertical displacement at the point Q is $u_Q^* = 2\ mm$ and the required force values are $F_Q = 4\ kN$, $F_Q = 5\ kN$, $F_Q = 6\ kN$. The influence functions for punch are calculated as $H_{i,i}^{(1)} = l_i / AE$ and for the beam according to Appendix A.

In the starting position of the punch, at $x_0 = 250$ mm, $l_c = L_4 - L_1 = 60\ mm$, the maximum stress is $\sigma_{\max} = 536\ MPa$. However, for the optimal solution the punch should be moved in the right direction. The optimal deflection form, initial gap form and $\sigma_{\max}$ are presented in Fig. 3 for three values of force F_Q. A special “second type iteration” method [5] was used for solution of these optimization problems. At the optimal solution the values of load F_0 and pressure $p_{\max}$, position length L_1, rigid body displacement λ_F^1 are collected in Table 1. The load factor efficiency now is $f_0 = F_0 / F_Q = \sim 3$.



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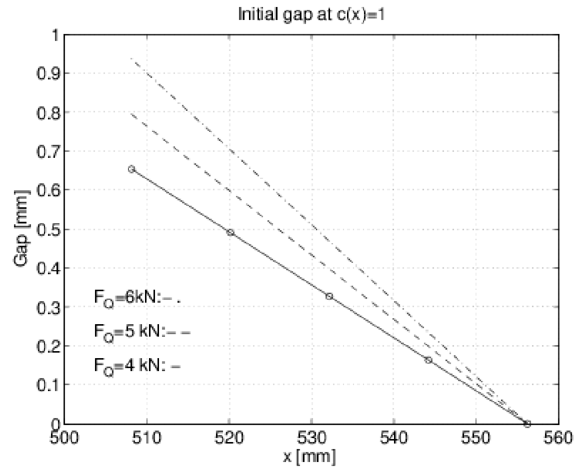

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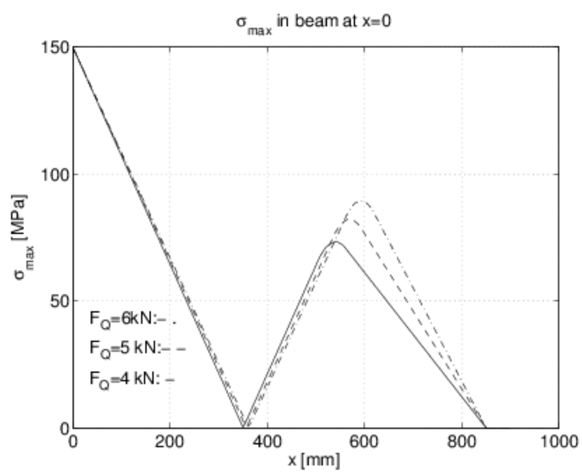


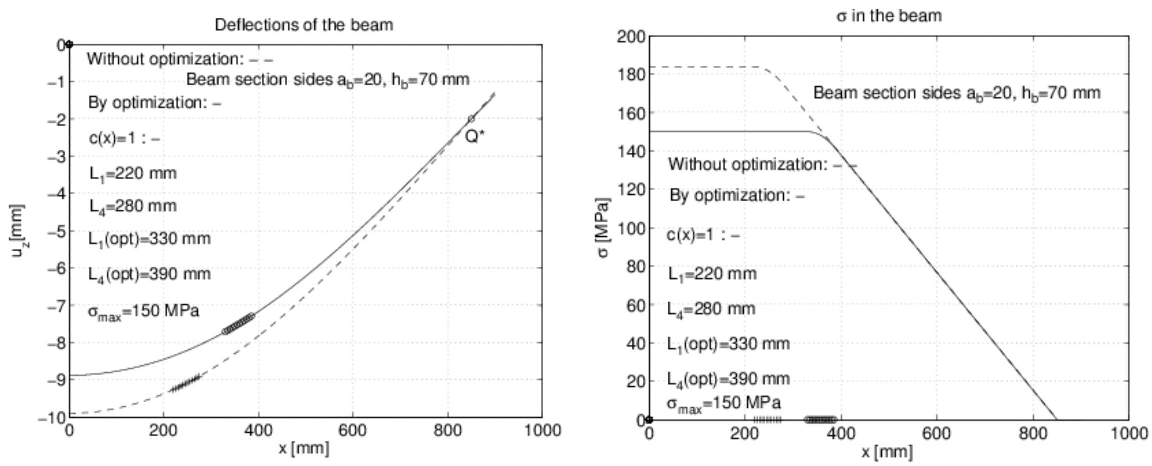
c)

Fig.3. Beam I deflection form for $c(x)=1$, a) for prescribed forces $F_Q = 4\,kN$, $F_Q = 5\,kN$, $F_Q = 6\,kN$, b) Initial contact gap, c) $\sigma_{\max}$ distribution along beam length. /Position of punch is denoted by o at $F_Q = 5\,kN$, see Fig. 3a/

Table 1. Punch loads, maximal contact pressures, punch positions, and rigid body displacements for optimal design solutions

$F_Q\ [kN]$	$F_0\ [kN]$	$p_{\max}\ [MPa]$	$L_1\ [mm]$ $x_0 = L_1 + l_c / 2$	$\lambda_F^1\ [mm]$
4	12.13	10.11	502.12	5.16
5	15.02	12.51	534.59	11.86
6	17.90	14.92	560.76	12.78

2.2 ***Example: beam II for the continuous punch action***: Geometric data and material parameters are the same before. It is also supposed that $c(x)=1$, that is contact pressure is constant, $p_n(x)=F_0/(b(L_4-L_1))=4.1666\,MPa$. The force at point Q is equal to $F_Q=5\,kN$, the required vertical displacement is $u_n^*=2\,mm$. The punch centre position is assumed at $x_0=250\ \ mm$ (for $L_1=220\,mm$, $L_4=280\,mm$). The plane stress state is assumed in the elastic punch . The normal displacement $u_n^{(1)}(x,p_n)$ was calculated by the finite element method, using *p*-version technique, Szabó (1991). Solving the contact problem, it is found $\lambda_F^{(1)}=9.26\,mm,\lambda_F^{(2)}=9.90\,mm$ and the beam maximal bending stress is $\sigma_{\max}=183.67\,MPa$. Assuming the admissible stress value $\sigma_u=150\,MPa$, the punch action must be moved in the right direction. Using the iteration method of Ref. [5] for solution of this optimization problem, we obtain the punch centre position at $x_0=360\,mm$, the contact zone length $l_c=60\,mm$ and the position factor $\xi_c=x_0/x_Q=0.4235$. The rigid body displacements are $\lambda_F^{(1)}=7.71\,mm,\lambda_F^{(2)}=8.88\,mm$. The deflection curves and the maximal normal stresses of initial and optimized designs are shown in Fig. 4.



a) b)

Fig. 4. a) Beam II deflection forms of the initial and optimized designs, b) maximal normal stresses along the beam axis x for two designs. The punch position is marked by o or +.

3. Control of beam deflection and slope by the action of two concentrated loads

Consider now the beam of Fig. 5 with the sliding support at left end, loaded by two concentrated loads $F_0^-=F_Q^-$ and $F_0^+=F_Q^+$ at the distances x_0^- and x_0^+. The specified contact force at Q located at the distance x_Q now is $F_Q=F_Q^-+F_Q^+$. At the support the transverse force equals $F_b=0$. Denote the load position factors by $\xi_0^-=x_0^-/x_Q$ and $\xi_0^+=x_0^+/x_Q$. The case of concentrated loads can be treated analytically and its solution provides an input to the analysis of distributed two punches action.

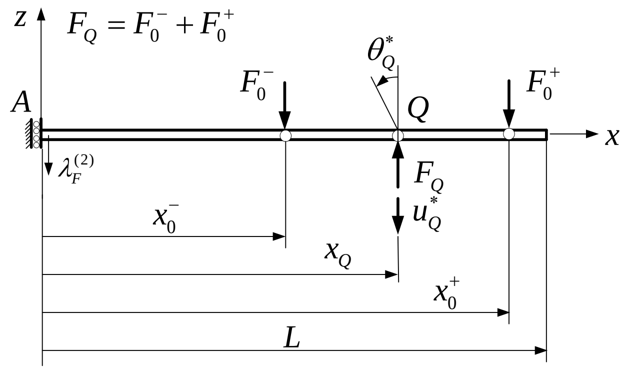


Fig. 5. Beam II is loaded by two forces F_0^-, F_0^+ inducing at the point Q the required force $F_Q = F_Q^- + F_Q^+$. The exact normal displacement value (deflection) at Q is required, $w = u_Q^*$, also the slope at this point is required, $\theta = dw/dx = \theta_Q^*$.

The normal deflection at the point Q (along –z-axis) due to force F_Q (can be expressed from the formulae of *Appendix A)* equals $w_Q = -\frac{F_Q}{3EI}x_Q^3 = -|w_Q|$. Deflection in the direction –z will be denoted by w. Consider first the case of single force action at $x_0^- \le x_Q$, or $\xi_0^- \le 1$. The deflection form now is expressed as follows

$$w^- = -\frac{F_Q^-}{2EI}(x_Q - x_0^-)x^2 + w_b^-, \quad 0 \le x \le x_0^-, \; w^- = -\frac{F_Q^-}{2EI}\left[\left[\; x_Q x - \left(x_0^-\right)^2 \;\right] x + \frac{1}{3}\left[\; \left(x_0^-\right)^3 - x^3 \;\right] \right] + w_b^-, \quad x_0^- \le x \le x_Q$$

$$w^- = -\frac{F_Q^-}{2EI}\left[\left[\; x_Q^2 - \left(x_0^-\right)^2 \;\right] x + \frac{1}{3}\left[\; \left(x_0^-\right)^3 - x_Q^3 \;\right] \right] + w_b^-, \quad x_Q \le x \tag{14}$$

For the single force action at $x_0^+ \ge x_Q$, or $\xi_0^+ \ge 1$, we have

$$w^+ = \frac{F_Q^+}{2EI}(x_0^+ - x_Q)x^2 + w_b^+, \quad 0 \le x \le x_Q, \quad w^+ = \frac{F_Q^+}{2EI}\left[(x_0^+ x - x_Q^2)x - \frac{1}{3}(x^3 - x_Q^3) \right] + w_b^+, \quad x_Q \le x \le x_0^+$$

$$w^+ = \frac{F_Q^+}{2EI}\left[(\left(x_0^+\right)^2 - x_Q^2)x - \frac{1}{3}(\left(x_0^+\right)^3 - x_Q^3) \right] + w_b^+, \quad x_0^+ \le x \tag{15}$$

When two forces load the beam, then

$$w = w^- + w^+, \; F_Q = F_Q^- + F_Q^+, \; w_b = w_b^- + w_b^+ \tag{16}$$

where w_b denotes the beam translation at the support.

Supposing the contact interaction load F_Q to be attained by the combined two forces action, we can write

$$F_Q = F_Q^- + F_Q^+ = k^-\left(u_Q^*\right)^m + k^+\left(u_Q^*\right)^m. \tag{17}$$

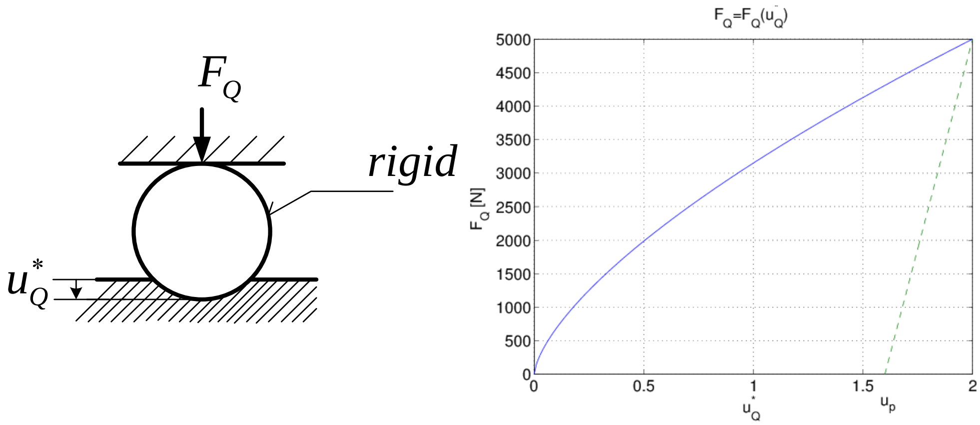


Fig.6. Contact of a rigid sphere with an elastic-plastic body: non-linear indentation function

$$F_Q = F_Q(u_Q^*) = k\left(u_Q^*\right)^m$$

Requiring $w = u_Q^*$ at Q, the beam translation w_b is then determined, thus

$$w_b = u_Q^* + \frac{k^-\left(u_Q^*\right)^m}{3EI} x_Q^3 \left[\ 1 - \frac{1}{2}\left(3\left(\xi_0^-\right)^2 - \left(\xi_0^-\right)^3\right)\ \right] - \frac{1}{2}\frac{k^+\left(u_Q^*\right)^m}{EI} x_Q^3 (\xi_0^+ - 1) \tag{18}$$

If the separate load action is assumed, the beam deflections at the points x_0^-, x_0^+ are

$$w_0^- = u_Q^* + \frac{k^-\left(u_Q^*\right)^m}{3EI} x_Q^3 \left[\ 1 - 3\left(\xi_0^-\right)^2 + 2\left(\xi_0^-\right)^3\ \right], \quad \xi_0^- \le 1 \tag{19}$$

and

$$w_0^+ = u_Q^* + \frac{k^+\left(u_Q^*\right)^m}{3EI} x_Q^3 \left[\ (\xi_0^+)^3 - 3\xi_0^+ + 2\ \right], \quad \xi_0^+ \ge 1 \tag{20}$$

At force $F_Q^- = k^-\left(u_Q^*\right)^m$ and $F_Q^+ = k^+\left(u_Q^*\right)^m$ deflection at the point Q is

$$\left|w_Q^{\mp}\right| = k^{\mp}\left(u_Q^*\right)^m x_Q^3 / 3EI \tag{21}$$

Using (13) and (14) we can derive the formulae for displacements at the points x_0^-, x_0^+ due to two forces

$$w_0^- = w_0^-(u_Q^*) = \frac{k^-\left(u_Q^*\right)^m}{3EI} x_Q^3 \frac{3}{2}\left[\ \left(\xi_0^-\right)^3 - \left(\xi_0^-\right)^2\ \right] + \frac{k^+\left(u_Q^*\right)^m}{3EI} x_Q^3 \frac{3}{2}\left(\xi_0^-\right)^2 (\xi_0^+ - 1) + w_b \tag{22}$$

$$w_0^+ = w_0^+(u_Q^*) = \frac{k^-\left(u_Q^*\right)^m}{3EI} x_Q^3 \frac{1}{2}\left[3\cdot\xi_0^+(\xi_0^-)^2 - 3\xi_0^+ - (\xi_0^-)^3 + 1\ \right] + \frac{k^+\left(u_Q^*\right)^m}{3EI} x_Q^3 \frac{1}{2}(1 + 2(\xi_0^+)^3 - 3\xi_0^+) + w_b \tag{23}$$

If the $k^{\mp} = k^{\mp}(\tau)$ are changing in time, then the forces $F_Q(\tau) = F_Q^-(\tau) + F_Q^+(\tau) = F_0$ and the deflection also varies in time, thus

$$w(\tau) = w^-(\tau) + w^+(\tau) \tag{24}$$

At the point Q the beam deflection is required to preserve the value $u_Q^*(\tau)$. This value is reached through the varying displacements $w_0^-(\tau), w_0^+(\tau)$. Alternatively, the force-displacement relation $F_Q = F_Q\left(u_Q^*\right)$ can be attained by one varying force action.

Consider now the advanced beam control by requiring the specified deflection and slope value to be preserved at the interaction point Q. Such control can be attained by the action of two punches at x_0^+ and x_0^-. Considering the concentrated load action, from Eqs. (13) and (14) we obtain the expression of slope value

$$w_Q^{'} = \frac{dw}{dx} = -\frac{F_Q^- x_Q^2}{2EI}\left(1-\left(\xi_0^-\right)^2\right) + \frac{F_Q^+ x_Q^2}{2EI} 2(\xi_0^+ - 1) = \theta_Q^* \tag{25}$$

Satisfying the condition $F_Q = F_0^- + F_0^+ = F_Q^- + F_Q^+$, from (24) the values of two loads per unit beam width are obtained, thus

$$F_0^- = \frac{2F_Q x_Q^2(\xi_0^+ - 1) - 2EI\theta_Q^*}{x_Q^2\left[2\xi_0^+ - \left(\xi_0^-\right)^2 - 1\right]}, \quad F_0^+ = \frac{F_Q x_Q^2\left[1-\left(\xi_0^-\right)^2\right] + 2EI\theta_Q^*}{x_Q^2\left[2\xi_0^+ - \left(\xi_0^-\right)^2 - 1\right]}, \tag{26}$$

Denoting the load fractions by $f_0^- = F_0^- / F_Q$ and $f_0^+ = F_0^+ / F_Q$, the diagram of evolution of the slope angle θ_Q^* on load fractions and their positions is presented in Fig 7. The relations (25) can now be written as follows

$$f_0^- = \frac{2(\xi_0^+ - 1) - 2\beta\theta_Q^*}{2\xi_0^+ - \left(\xi_0^-\right)^2 - 1}, \quad f_0^+ = \frac{\left[1-\left(\xi_0^-\right)^2\right] + 2\beta\theta_Q^*}{2\xi_0^+ - \left(\xi_0^-\right)^2 - 1}, \quad \beta = \frac{EI}{F_Q x_Q^2} \tag{27}$$

If we take account influence on pin (see Fig.8) for deflection and slope, $EI_{\mathrm{mod}} = EI - sF_Q x_Q$ will taken instead of EI.

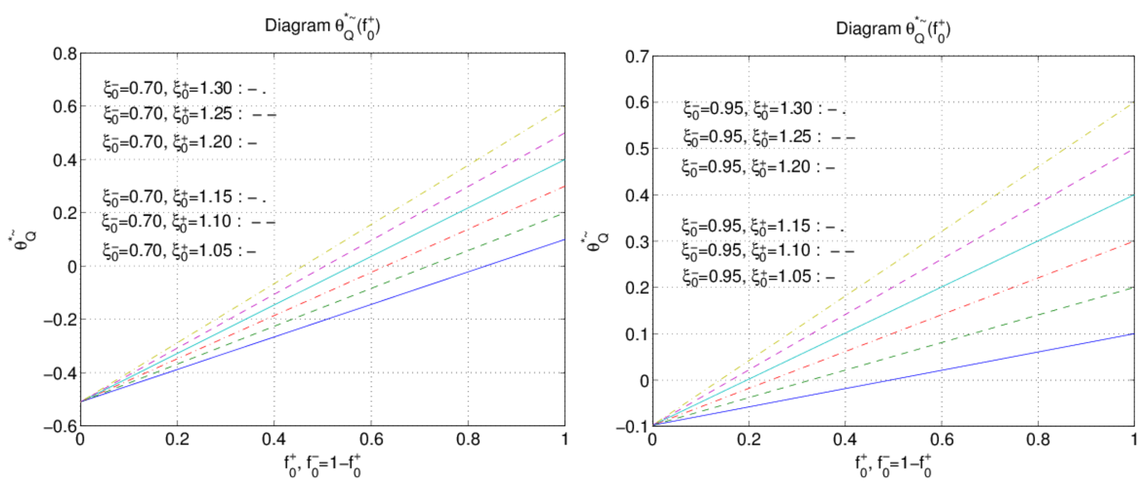


Fig 7. Dependence of the beam slope $\theta_Q^{*\sim} = 2\beta\theta_Q^*$ on two load fractions f_0^-, f_0^+ their positions ξ_0^-, ξ_0^+ and beam stiffness-load ratio $\beta = EI / F_Q x_Q^2$.

4. Control of displacement and slope in a beam with a transverse tool at point *Q*

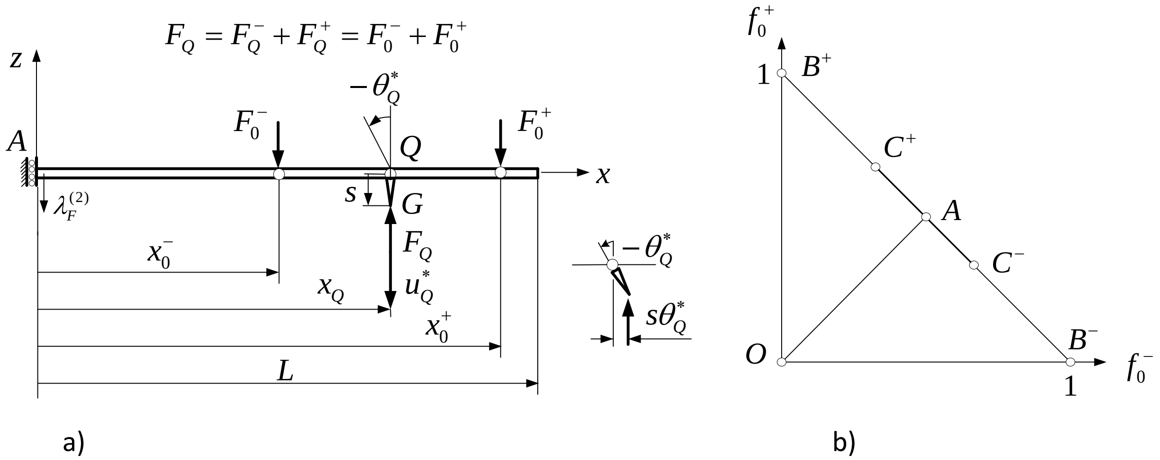


Fig. 8. Beam II with a rigid transverse tool at Q loaded by two forces F_0^-, F_0^+ such that $F_0^- + F_0^+ = F_Q$. The control of displacement u_Q^* and slope θ_Q^* at Q provides the control of tool end displacement.

Consider now the beam with transversely attached rigid pin of length *s*, Fig. 8. Now the control of beam is aimed to induce a required trajectory in the plane *x*-*z*. Assume that first the normal deflection u_Q^* is attained at vanishing slope $\theta_Q^* = 0$ and next the end of pin executes reciprocal translation of amplitude $2\tilde{a}$ along the *x*-axis at fixed value of deflection. The derived formulae (25) or (26) allow us to execute such pin motion by varying two loads. First, setting θ_Q^*=0 in (25) the values of F_0^- and F_0^+ are obtained and the value of F_Q assures the transverse displacement value at *Q*. The ratio of forces equals

$$\frac{f_0^-}{f_0^+} = \frac{F_0^-}{F_0^+} = \frac{2(\xi_0^+ - 1)}{1-\left(\xi_0^-\right)^2} \quad , \qquad f_0^- + f_0^+ = 1 \qquad , \tag{28}$$

In the load diagram, Fig. 8b, the initial force state is represented by the point *A*. Next, the loads vary following the loading path $f_0^- + f_0^+ = 1$ in the force plane. For the assumed symmetric sliding of the pin end relative to the initial position, the force reversal states are represented by the points C^+ and C^- in Fig. 8b. Note that $AC^- / AC^+ = AB^- / AB^+$ Then the maximal and minimal values of the slope are reached at B^- and B^+ for the single load actions. From (26) it follows that

$$\left(\theta_Q^*\right)^- = -\frac{f_0^-\left[2\xi_0^+ - \left(\xi_0^-\right)^2 - 1\right] - 2(\xi_0^+ - 1)}{2\beta}, \quad \left(\theta_Q^*\right)^+ = \frac{f_0^+\left[2\xi_0^+ - \left(\xi_0^-\right)^2 - 1\right] - \left[1-\left(\xi_0^-\right)^2\right]}{2\beta} \tag{29}$$

and the total slope equals $\theta_Q^* = \left(\theta_Q^*\right)^- + \left(\theta_Q^*\right)^+$ The extreme slope values are obtained for $f_0^- = 1$, $f_0^+ = 0$ and $f_0^+ = 1$, $f_0^- = 0$, thus

$$\left(\theta_Q^*\right)_{\min} = -\frac{1-\left(\xi_0^-\right)^2}{2\beta}, \qquad \left(\theta_Q^*\right)_{\max} = \frac{\xi_0^+ - 1}{\beta} \tag{30}$$

The related tool end displacements are $\Delta x^{-} = -\left(\theta_Q^{*}\right)_{\min} s$, $\Delta x^{+} = -\left(\theta_Q^{*}\right)_{\max} s$ and the maximal translation amplitude is $\left(\Delta x\right)_a = \left[2\xi_0^{+} - 1 - \left(\xi_0^{-}\right)^2\right] s / \beta$

When the specified tool sliding amplitude is less than the maximal value (30), the loading program is executed along the portion of the loading path shown in Fig. 8b. Non-linear effects related to large beam deflections, large slope variation at Q and the effect of contact friction force are not considered here. The slope dependence on the parameter β is shown in Fig. 9 and the deflection curve is presented in Fig. 10.

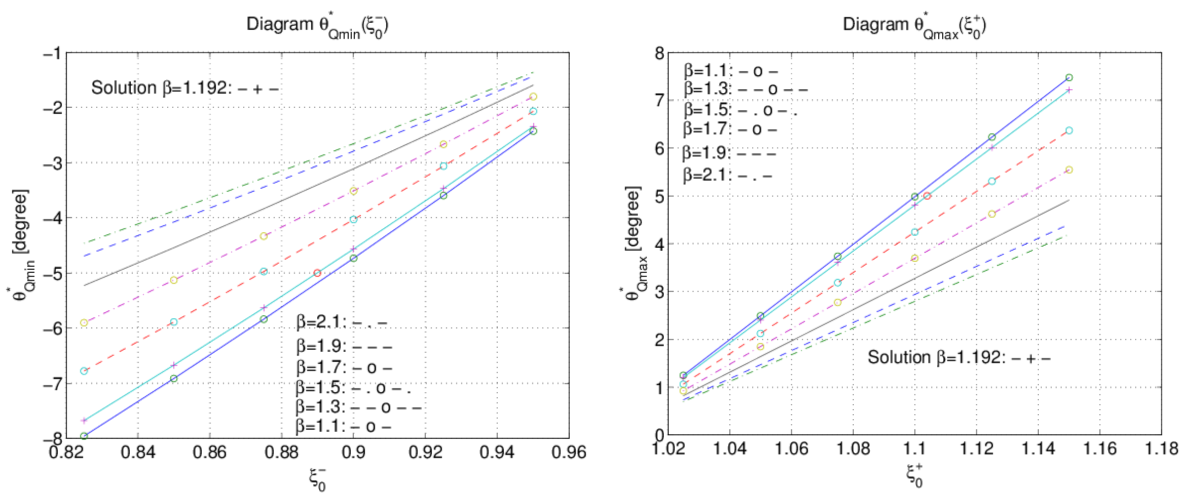


Fig. 9. Minimum and maximum of the slope for varying parameter β .

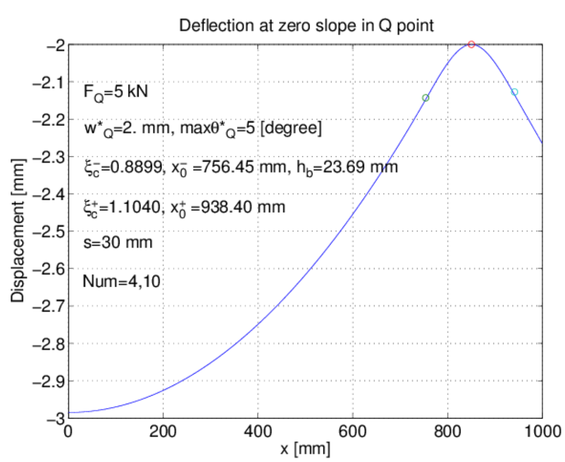


Fig. 10. Deflection of the beam II for loads $F_0^{-} = F_0^{+}$, $f_0^{-} = f_0^{+} = 0.5$, $\beta = 1.2271$, $\beta_{\text{mod}} = 1.1918$

The formulae presented are valid also for the case $\xi_c^{-} = 0$, when the load F_0^{-} is applied at the sliding support. A similar design is used in the atomic force microscope, where a cantilever beam with a sliding support at the left end and a sharp transverse pin at the right end are used. The load is then applied at the left support and its translation is controlled in order to generate proper contact force of pin. The present design is more complex requiring the pulsating load application of two punches in order to generate the reciprocal sliding motion of pin. Such design and punch action can be used in developing a wear testing apparatus at both micro and macro-scales.

Conclusions

The optimal design of punch action in order to control normal displacement at a loaded boundary point in a structural element has been discussed in the paper and illustrated by the specific examples of beam deflection control. A new class of optimization problems has been analysed, when at one boundary point both displacement and force are specified. It has been shown that support constraint can affect essentially the punch load and the beam deflected form. The problem formulation was also extended to control both the deflection and its slope at a loaded boundary by the action of two punches. For varying punch loads the control was aimed to induce reciprocal tool sliding on the contact surface.

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References

1. Banichuk, N.V.: Introduction to Optimization of Structures, Springer, London, 2011.
2. Mróz, Z. and Bojczuk, D.: Shape and topology sensitivity analysis and its application to structural design**,** Archive of Applied Mechanics, 82(10-11), 1541-1555 (2012).
3. Goryacheva, I.G.: Contact Mechanics in Tribology, Kluwer Academic Publishers, Dordrecht, 1998.
4. Páczelt, I., and Mróz, Z.: Solution of wear problems for monotonic and periodic sliding with p-version of the finite element method, Comput. Methods Appl. Mech. Engng. 249-252, 75-103, (2012).
5. Páczelt, I., Baksa, A. and Mróz, Z.: Contact optimization problems for stationary and sliding conditions, Springer International Publishing Switzerland 2016, P. Neittaanmaki et al. (eds.), Mathematical Modelling and Optimization of Complex Structures, Computational Methods in Applied Sciences *40*, 281-312, DOI 10.1007/978-3-319-23564-6_16.
6. Páczelt, I.: Some optimization problems of contact bodies within the linear theory of elasticity, In S. Nemat-Nasser (ed.) Variational methods in the Mechanics of Solids, pp. 349-356, Pergamon Press, Oxford,1980.
7. Páczelt, I.: Iterative methods for solution of contact optimization problems, Arch. Mech., 52, 4-5, 685-711, (2000).
8. Wriggers, P.: Computational Contact Mechanics, Wiley, New York, 2002.
9. Szabó, B. and Babuska, I.: Introduction to finite element analysis, Wiley-Intersience, New York, 1991.
10. Monkman G, Hesse S, Steinmann R, Schunk H: Robot grippers. Wiley-VCH Berlin, 2006.

Appendix A

Using Betti theorem, the influence (Green) function for cantilever beam (Fig. 11) has the form

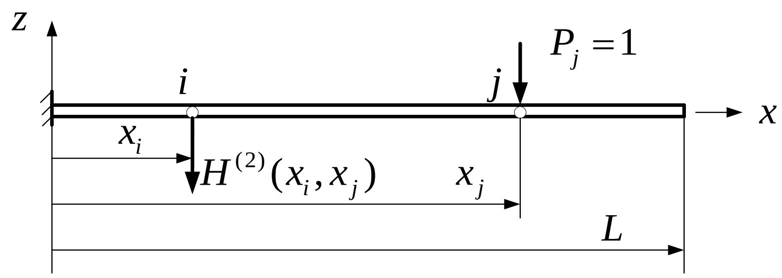

Fig. 11. Load for calculation of the Influence function of the cantilever beam

$$H^{(2)}(x_i, x_j) = H^{(2)}_{i,j} = \frac{1}{6EI}\left\{(3x_j x_i^2 - x_i^3) + \left\langle x_i - x_j \right\rangle^3\right\},$$

where $\left\langle x_i - x_j \right\rangle^3 = \begin{cases} (x_i - x_j)^3, if \;\; x_i > x_j \\ 0, \qquad\quad if \;\; x_i \le x_j \end{cases}$, I -is the inertia moment of cross section, E is the Young modulus.