EXISTENCE AND UNIQUENESS OF POTENTIAL FLOW SOLUTIONS FOR ADJUSTABLE STRAIGHT CASCADE OF AEROFOILS

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(Received 5 June 1999)

Abstract. Martensen proved that the only nontrivial solution of the homogeneous adjoint equation to the Fredholmian integral equation of the second kind, written for the potential flow around a cascade of aerofoils, is the constant function. In this paper a simple proof is given to the theorem, furthermore, it is extended for the related singular integral equation of the first kind and for the adjustable cascade of aerofoils.

1. Introduction

Impellers or runners of axial-flow turbomachines, e.g. pumps, turbines and fans, usually have adjustable blades in order to attain high efficiency over a wide range of flow rate domain. Devices for prewhirl control of pumps and the wicket gates in turbines are both equipped with adjustable blades. By turning these blades a new arrangement is created. A designer faces a similar problem when choosing the number and chord length of blades, and hence the cascade solidity, to give optimal blade load. It often happens that three or four versions of impellers with different number of blades are produced in order to meet the requirements. The proper version is finally chosen by experimental testing. It would be much more economical, however, to study the effect of cascade solidity on the hydraulic parameters of turbomachines while designing the bladings, i.e. before manufacturing.

The flow around the blading of axial-flow turbomachines can usually be modelled with acceptable accuracy by using a straight cascade of blades. There are several methods available for the determination of an attached flow through a straight cascade of foils assuming incompressible inviscid fluid [1–7].

When modifying the cascade geometry by adjusting the blades, the usual procedure is to repeat most of the computational work. This paper suggests a method by which we can avoid the necessity of repeated applications of the direct computational procedure for each cascade geometry. It is also an extension of the theory published earlier [8], and is a more concise version of the author's thesis [9]. We are not going to deal with computational details here, the objective of this paper is to investigate existence and unicity of the solution.

2. Derivation of the governing equations

In this paper we deal with the frictionless solenoidal attached flow of an incompressible fluid around the blades of a straight cascade. The equation of continuity and that of expressing that flow is vortex free can be written as follows:

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0, \tag{1}$$

$$\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = 0. \tag{2}$$

Here w_x and w_y are the x and y components of the relative velocity vector measured in the co-ordinate system fixed to the rotary or stationary blades (see Fig. 1). It is easy to see that (1) and (2) represent the Cauchy–Riemann equations for the conjugate complex velocity

$$\overline{w} = w_x - \imath w_y, \tag{3}$$

where $\imath$ is the imaginary unit, $\imath = \sqrt{-1}$.

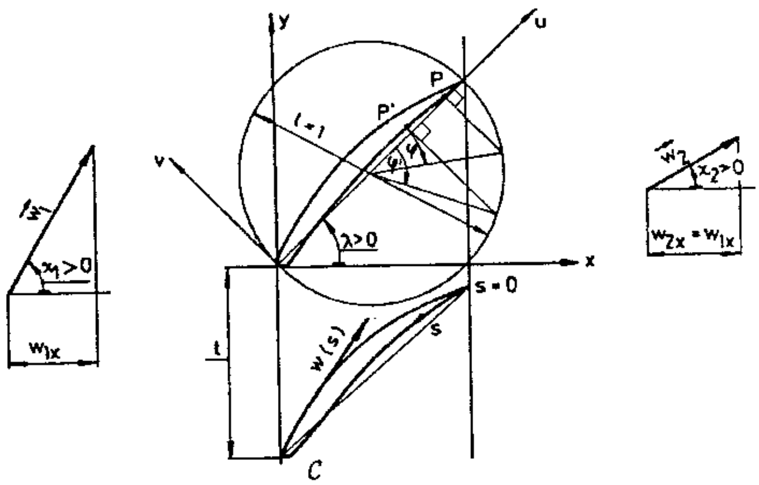

Fig. 1. Notations for a straight cascade of aerofoils

Since the components of the conjugate complex velocity satisfy the Cauchy–Riemann equations, $\overline{w}$ is an analytic function of the complex variable

$$z = x + \imath y. \tag{4}$$

Hence Cauchy's integral theorem and Plemelj's formulae [11] can be applied to the conjugate complex velocity which finally results in (see [2, 3, 9])

$$\overline{w}(\varsigma) + \frac{1}{2\pi}\oint_C K(\varsigma,\varsigma')\overline{w}(\varsigma')d\varsigma' = 2\overline{w}_\infty, \tag{5}$$

where ς and ς' are complex variables along contour C of the foil (see Fig. 1), and $\overline{w}_\infty$ is the mean value of upstream and downstream conjugate complex velocities, $\overline{w}_1$ and $\overline{w}_2$,

$$\overline{w}_\infty = \frac{1}{2}(\overline{w}_1 + \overline{w}_2). \tag{6}$$

In (5) ′ refers to the variable point of integration. If $\varsigma' \neq \varsigma$ the kernel function can be written as

$$K(\varsigma,\varsigma') = \frac{\pi}{t}\coth\frac{\pi}{t}(\varsigma-\varsigma'), \tag{7}$$

where t is the blade spacing. It can easily be seen that (5) is a complex variable integral equation of the second kind for the conjugate complex velocity around contour C. In the knowledge of $\overline{w}(\varsigma)$ along C, the conjugate complex velocity $\overline{w}(z)$ at an arbitrary inner point of the flow can also be determined [3, 9], although we do not intend to deal with this problem here.

Equation (5) can be divided into two scalar equations: one in the tangential and the other in the normal direction to the aerofoil. This results in two real integral equations for the contour velocity as a function of arc length s. Then we will introduce polar angle φ, shown in Fig. 1, instead of arc length s, and the transformed dimensionless velocity $q(\varphi)$ defined by (see [3, 9])

$$w_t(s)ds = q(\varphi)w_{1x}d\varphi, \tag{8}$$

where w_t is the tangential component of the velocity along C and w_{1x} is the x component of the upstream velocity (see Fig. 1). After introducing this transformation into our equations, the following two integral equations are obtained

$$q(\varphi) + \frac{1}{\pi}\int_0^{2\pi} K(\varphi,\varphi')q(\varphi')d\varphi' = f(\varphi), \tag{9}$$

$$\frac{1}{\pi}\int_0^{2\pi} H(\varphi,\varphi')q(\varphi')d\varphi' = h(\varphi), \tag{10}$$

where kernel functions $K(\varphi,\varphi')$ and $H(\varphi,\varphi')$ for $\varphi' \neq \varphi$

$$K(\varphi,\varphi') = \frac{\pi}{t}\frac{\dot{y}(\varphi)\sinh\frac{2\pi}{t}[x(\varphi)-x(\varphi')] - \dot{x}(\varphi)\sin\frac{2\pi}{t}[y(\varphi)-y(\varphi')]}{\cosh\frac{2\pi}{t}[x(\varphi)-x(\varphi')] - \cos\frac{2\pi}{t}[y(\varphi)-y(\varphi')]} + \frac{\pi}{t}\dot{y}(\varphi) \tag{11}$$

and

$$H(\varphi,\varphi') = \frac{\pi}{t}\frac{\dot{x}(\varphi)\sinh\frac{2\pi}{t}[x(\varphi)-x(\varphi')] + \dot{y}(\varphi)\sin\frac{2\pi}{t}[y(\varphi)-y(\varphi')]}{\cosh\frac{2\pi}{t}[x(\varphi)-x(\varphi')] - \cos\frac{2\pi}{t}[y(\varphi)-y(\varphi')]} + \frac{\pi}{t}\dot{x}(\varphi). \tag{12}$$

Carrying out the limiting process $\varphi' \to \varphi$ in Eqs. (11) and (12) while repeatedly applying the l' Hospital's rule yields

$$\lim_{\varphi'\to\varphi} K(\varphi,\varphi') = \frac{1}{2}\frac{\ddot{y}(\varphi)\dot{x}(\varphi) - \ddot{x}(\varphi)\dot{y}(\varphi)}{[\dot{x}(\varphi)]^2 + [\dot{y}(\varphi)]^2} + \frac{\pi}{t}\dot{y}(\varphi), \tag{13}$$

$$\lim_{\varphi'\to\varphi}(\varphi-\varphi')H(\varphi,\varphi') = 1. \tag{14}$$

The right-hand sides (RHSs) of Eqs. (9) and (10) can be written as follows

$$f(\varphi) = 2\dot{y}(\varphi)\tan\chi_1 + 2\dot{x}(\varphi), \tag{15}$$

$$h(\varphi) = 2\dot{x}(\varphi)\tan\chi_1 - 2\dot{y}(\varphi). \tag{16}$$

In these equations $x(\varphi)$, $y(\varphi)$, $\dot{x}(\varphi)$, $\dot{y}(\varphi)$, $\ddot{x}(\varphi)$ and $\ddot{y}(\varphi)$ are profile co-ordinates and their derivatives with respect to φ.

It can easily be seen from the above equations that in the limiting case when φ' tends to φ, kernel function $K(\varphi,\varphi')$ is bounded but kernel function $H(\varphi,\varphi')$ has a first order singularity. Hence (9) is a Fredholmian integral equation of the second kind and (10) is a singular integral equation of the first kind.

It is known that equations for potential flow around a single aerofoil can be obtained if we carry out the limiting case when the blade spacing t tends to infinity in the above equations, see [3, 9]. In this case the equations are similar to (9)-(16) but the kernel functions are different. Due to lack of space we do not intend to deal with this problem.

In principle any of the two Eqs. (9) and (10) can be used for the solution of flow around a cascade of aerofoils or that of the single aerofoil. However, as has already been stated (see [6]) it is more advisable to choose integral equations of the second kinds for numerical treatment, i.e. (9). The author's own experiences also confirmed this statement.

3. Existence and uniqueness of the solution

Martensen [1] has proved that the homogeneous adjoint equation to (9)

$$\beta(\varphi) + \frac{1}{\pi}\int_0^{2\pi} K(\varphi', \varphi)\beta(\varphi')d\varphi' = 0 \tag{17}$$

has a $\beta(\varphi) = const$ nontrivial solution, hence the following relation holds true for kernel function $K(\varphi, \varphi')$:

$$1 + \frac{1}{\pi}\int_0^{2\pi} K(\varphi, \varphi')d\varphi' = 0. \tag{18}$$

Martensen's proof is a very complicated one. Let us suggest a simpler proof.

As it is known [10], the kernel function has first order singularity as ς' tends to ς, and can be written

$$\frac{\pi}{t}\coth\frac{\pi}{t}(\varsigma - \varsigma') = \frac{1}{\varsigma - \varsigma'} + \frac{1}{3}\left(\frac{\pi}{t}\right)^2(\varsigma - \varsigma') + o(3), \tag{19}$$

where o stands for the order of magnitude. Hence

$$G(\varsigma, \varsigma') = \frac{\pi}{t}\coth\frac{\pi}{t}(\varsigma - \varsigma') - \frac{1}{\varsigma - \varsigma'} \tag{20}$$

is an analytic function along curve C since the principal part of its Laurent series vanishes [11]. Cauchy's theorem states that for any analytic function

$$\oint_C G(\varsigma, \varsigma')d\varsigma = 0, \tag{21}$$

where C is an arbitrary non-intersecting rectlinear closed curve bounding a singly connected domain [11]. According to Cauchy's integral formula

$$\oint_C \frac{1}{\varsigma - \varsigma'}d\varsigma = -\pi\imath,$$

when ς' lies on such a curve C the integral along which is taken in clockwise direction. Combination of this latter equation with (20) and (21) yields

$$\pi\imath + \oint_C \frac{\pi}{t}\coth\frac{\pi}{t}(\varsigma - \varsigma')d\varsigma = 0. \tag{22}$$

By introducing polar angle φ instead of arc length s along contour C and separating the equation into real and imaginary parts we obtain

$$\pi + \int_0^{2\pi} K(\varphi, \varphi')d\varphi = 0, \tag{23}$$

$$\int_0^{2\pi} H(\varphi,\varphi')d\varphi = 0. \tag{24}$$

Equation (23) is π times of Eq. (18) to be proved. On the other hand (24) expresses the fact that the constant function is also a nontrivial solution of the homogeneous adjoint equation to Eq. (10). With this Martensen's theorem is proved.

As the homogeneous adjoint equation (17) has nonzero solutions, according to Fredholm's theorems the original integral equation (9) can have solutions only and if only the function on the RHS of (9) is orthogonal to the eigenfunctions of the homogeneous adjoint equations. In this case it has infinite number of solutions. Since these eigenfunctions are the constant functions, the orthogonality condition can be written as

$$\int_0^{2\pi} f(\varphi)d\varphi = 0. \tag{25}$$

It can easily be seen that (25) holds true since

$$\oint_C f(\varphi)d\varphi = 2\tan\chi_1 \int_0^{2\pi} \frac{dy}{d\varphi}d\varphi + 2\int_0^{2\pi} \frac{dx}{d\varphi}d\varphi = 2\tan\chi_1 \oint_C dy + 2\oint_C dx = 0. \tag{26}$$

Consequently, integral equation (9) has an infinite number of different solutions differing from each other only by additional constants. The real solution can be chosen from this infinite set of solutions by applying the Kutta condition. This condition expresses the fact that the trailing edges of the blades are unloaded.

In case of a single aerofoil the proof can be done in a similar way.

4. Adjustable cascades

So far the stagger angle and blade spacing were fixed values. We shall consider these quantities now as independent variables, and study their effects upon the cascade flow, while the blade shape is kept unchanged. In this way, the turning of the blades of a rotor corresponding to a change of flow rate can be modelled. We note, on the other hand, that a knowledge of the effect of blade spacing t on the flow may be of value when finding the proper value of blade load. Let us see Eq. (9) in more detail:

$$q(\varphi;\lambda,t) + \frac{1}{\pi}\int_0^{2\pi} K(\varphi,\varphi';\lambda,t)q(\varphi';\lambda,t)d\varphi' = f(\varphi;\lambda). \tag{27}$$

Since the procedures built up for investigating the effects of λ and t upon the cascade flow are very similar in nature, we introduce a new variable ε which can represent either λ or t. It seems straightforward to expand Eq. (27) into Taylor series with respect to ε around its fixed value ε_0. We have

$$\sum_{i=0}^{\infty} \frac{\partial^i q}{\partial\varepsilon^i}\frac{(\varepsilon-\varepsilon_0)^i}{i!} + \frac{1}{\pi}\int_0^{2\pi} \sum_{i=0}^{\infty} \frac{\partial^i K}{\partial\varepsilon^i}\frac{(\varepsilon-\varepsilon_0)^i}{i!} \sum_{i=0}^{\infty} \frac{\partial^i q}{\partial\varepsilon^i}\frac{(\varepsilon-\varepsilon_0)^i}{i!} d\varphi' = \sum_{i=0}^{\infty} \frac{\partial^i f}{\partial\varepsilon^i}\frac{(\varepsilon-\varepsilon_0)^i}{i!}. \tag{28}$$

The derivatives in (28) are to be taken at $\varepsilon = \varepsilon_0$.

We note that the mutual effects of λ and t can also be investigated. In this case, Eq. (27) should be expanded into Taylor's series with respect to the two independent variables λ and

t around their fixed values λ_0 and t_0. The author derived the basic equations for this case, too. This method, however, unduly increases the number of equations to be solved and the subsequent numerical work. On the other hand, the separate study of the effects of the single parameters λ and t upon the flow is usually more important for the users. That is the reason why we do not intend to present that problem in this paper.

Let us return to Eq. (28). By separating the different powers of $(\varepsilon-\varepsilon_0)$, this equation can be resolved into an infinite set of integral equations for the derivatives of the transformed velocity q with respect to ε. The first few equations of this set are as follows:

$$\frac{\partial^i q(\varphi;\varepsilon)}{\partial\varepsilon^i}+\frac{1}{\pi}\int_0^{2\pi}K(\varphi,\varphi';\varepsilon_0)\frac{\partial^i q(\varphi';\varepsilon)}{\partial\varepsilon^i}d\varphi'=F_i(\varphi;\varepsilon_0)\quad(i=0,1,2,3,\ldots),$$

where

$$F_0(\varphi;\varepsilon_0)=f(\varphi;\varepsilon_0),$$

$$F_1(\varphi;\varepsilon_0)=\frac{\partial f(\varphi;\varepsilon)}{\partial\varepsilon}-\frac{1}{\pi}\int_0^{2\pi}\frac{\partial K(\varphi,\varphi';\varepsilon)}{\partial\varepsilon}q(\varphi';\varepsilon_0)d\varphi',$$

$$F_2(\varphi;\varepsilon_0)=\frac{\partial^2 f(\varphi;\varepsilon)}{\partial\varepsilon^2}-\frac{1}{\pi}\int_0^{2\pi}\left[\frac{\partial K^2(\varphi,\varphi';\varepsilon)}{\partial\varepsilon^2}q(\varphi';\varepsilon_0)+2\frac{\partial K(\varphi,\varphi';\varepsilon)}{\partial\varepsilon}\frac{\partial q(\varphi';\varepsilon)}{\partial\varepsilon}\right]d\varphi',$$

$$F_3(\varphi;\varepsilon_0)=\frac{\partial^3 f(\varphi;\varepsilon)}{\partial\varepsilon^3}-\frac{1}{\pi}\int_0^{2\pi}\left[\frac{\partial K^3(\varphi,\varphi';\varepsilon)}{\partial\varepsilon^3}q(\varphi';\varepsilon_0)+3\frac{\partial^2 K(\varphi,\varphi';\varepsilon)}{\partial\varepsilon^2}\frac{\partial q(\varphi';\varepsilon)}{\partial\varepsilon}\right]d\varphi'-$$

$$-\frac{3}{\pi}\int_0^{2\pi}\frac{\partial K(\varphi,\varphi';\varepsilon)}{\partial\varepsilon}\frac{\partial^2 q(\varphi';\varepsilon)}{\partial\varepsilon^2}d\varphi',\qquad\text{etc.}\tag{29}$$

The different orders of derivatives in Eqs. (29) are to be taken at $(\varepsilon-\varepsilon_0)$. All of these equations have the same kernel and their RHSs contain solutions of the preceding equations and the derivatives of the kernel function. The first equation in (29) is the one relating to a cascade with fixed value of λ and t. We note that the 0th derivative of a function is the function itself.

Let us investigate the existence and uniqueness of the solutions of system (29). It can easily be seen that every equation in the set (29) has the same structure as (9), which relates to a straight cascade with fixed values of λ and t. Hence the homogeneous adjoint equations to Eqs. (29) coincide with (17). That is why the constant function is a nonzero solution of each of these adjoint equations. According to Fredholm's theorems the original integral equations (29) can have solutions only in the case if the RHSs of Eqs. (29) are orthogonal to the eigenfunctions (here constant) of the homogeneous adjoint equations. The orthogonality condition can be written as

$$\int_0^{2\pi}F_i(\varphi;\varepsilon)d\varphi=0\quad(i=0,1,2,\ldots).\tag{30}$$

By substituting the RHSs of Eqs. (29) into (30), two types of conditions are obtained:

$$\int_0^{2\pi}\frac{\partial^i f(\varphi;\varepsilon)}{\partial\varepsilon^i}d\varphi=0\quad(i=0,1,2,\ldots),\tag{31}$$

$$\int_0^{2\pi}\int_0^{2\pi}\frac{\partial^i K(\varphi,\varphi';\varepsilon)}{\partial\varepsilon^i}\frac{\partial^j q(\varphi';\varepsilon)}{\partial\varepsilon^j}d\varphi'd\varphi=0\quad(i,j+1=1,2,3,\ldots).\tag{32}$$

The derivatives in these equations are to be taken at $(\varepsilon - \varepsilon_0)$.

Let us see (31) first. Equation (15) shows that

$$f(\varphi) = 2\dot{y}(\varphi;\lambda)\tan\chi_1 + 2\dot{x}(\varphi;\lambda). \tag{33}$$

Looking at Fig. 1, relations between x, y and u, v can easily be derived in the form

$$x(\varphi;\lambda) = u(\varphi)\cos\lambda - v(\varphi)\sin\lambda, \qquad y(\varphi;\lambda) = u(\varphi)\sin\lambda + v(\varphi)\cos\lambda.$$

The differentiation of these quantities with respect to λ and polar angle φ yields

$$\frac{\partial\dot{x}(\varphi;\lambda)}{\partial\lambda} = -\dot{y}(\varphi;\lambda), \qquad \frac{\partial\dot{y}(\varphi;\lambda)}{\partial\lambda} = \dot{x}(\varphi;\lambda). \tag{34}$$

Application of (33) and (34) gives

$$\frac{\partial^{4i} f(\varphi;\lambda)}{\partial\lambda^{4i}} = -\frac{\partial^{4i+2} f(\varphi;\lambda)}{\partial\lambda^{4i+2}} = f(\varphi;\lambda) \quad (i = 0,1,2,...),$$

$$\frac{\partial^{4i+1} f(\varphi;\lambda)}{\partial\lambda^{4i+1}} = -\frac{\partial^{4i+3} f(\varphi;\lambda)}{\partial\lambda^{4i+3}} = 2\dot{x}(\varphi;\lambda)\tan\chi_1 - 2\dot{y}(\varphi;\lambda) \quad (i = 0,1,2,...). \tag{35}$$

Since function f is independent of the blade spacing t, it holds true for arbitrary positive integer i that

$$\frac{\partial^i f}{\partial t^i} = 0 \quad (i = 1,2,3,...). \tag{36}$$

Bearing in mind (25), (33) and (34), it can easily be seen that (31) holds true for arbitrary value of i.

Let us investigate condition (32) now. The order of integration and derivation can be changed in the equation since the variables are independent of each other. Hence,

$$\int_0^{2\pi} \left\{ \frac{\partial^i}{\partial\varepsilon^i} \left[\int_0^{2\pi} K(\varphi,\varphi';\varepsilon)d\varphi \right] \right\} \frac{\partial^j q(\varphi';\varepsilon)}{\partial\varepsilon^j} d\varphi' = 0 \quad (i, j+1 = 1,2,3,...). \tag{37}$$

The derivatives here are to be taken at $(\varepsilon - \varepsilon_0)$. Bearing in mind (18), it can easily be seen that the value of the integral in the bracket is equal to $(-\pi)$, so its arbitrary orders of derivatives $(i = 1,2,...)$ vanish. Hence, Eqs. (37) and (32) hold true for all the possible cases. Since orthogonality conditions expressed by Eqs. (30) are fulfilled, every equation of (29) has an infinite number of different solutions. The real solution can be chosen by applying the Kutta condition.

In the knowledge of the solutions of Eqs. (29) belonging to $\varepsilon = \varepsilon_0$

$$q(\varphi;\varepsilon_0), \quad \frac{\partial q(\varphi;\varepsilon)}{\partial\varepsilon}, \quad \frac{\partial^2 q(\varphi;\varepsilon)}{\partial\varepsilon^2}, \quad \frac{\partial^3 q(\varphi;\varepsilon)}{\partial\varepsilon^3},$$

the transformed contour velocity belonging to parameter ε (λ or t) can be obtained:

$$q(\varphi;\varepsilon) = \sum_{i=0}^{\infty} \frac{\partial^i q(\varphi;\varepsilon)}{\partial\varepsilon^i} \frac{(\varepsilon - \varepsilon_0)^i}{i!}. \tag{38}$$

The derivatives in (38) are to be taken at $\varepsilon = \varepsilon_0$.

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