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# NUMERICAL METHOD FOR CALCULATING THE FLOW AROUND A CASCADE OF AEROFOILS

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## SUMMARY

A FORTRAN 1900 computer program has been written that gives the solution of the 2-D incompressible, inviscid flow problem for a rotating cascade of blades on the mean surface of a stream channel with constant width. The program is based on a known solution of the task. The paper includes the equations to be solved, the applied numerical methods, the flow chart of the computer program and sample calculations in comparison with experimental test and classical solutions.

### 1. DESCRIPTION OF METHOD AND BASIC EQUATIONS

There are a few solutions to determine the stream pattern around a cascade of foils on a surface of revolution assuming frictionless fluid [1],[4],[7],[10]. More known papers are for straight cascades to be studied when preparing a computer program for the more general case [5],[6],[9].

A full development of the basic theory is described in [7]. For the time being we take the special case when the channel width is constant. In the paper mentioned the arc length coordinate is used, which is to be transformed to a polar coordinate system by

$$\frac{w(s)}{w_{\infty}} ds = w(\varphi) l d\varphi \quad (1)$$

/see Fig.1/. With this the integral equation concerning the relative velocity  $w(\varphi)$  to be solved is as follows:

$$w(\varphi) + \frac{1}{\pi} \int_0^{2\pi} \mathcal{K}(\varphi, \varphi') w(\varphi') d\varphi' = F(\varphi) \quad (2)$$

where the kernel function is

$$\mathcal{K}(\varphi, \varphi') = \frac{\dot{y}(\varphi) \sinh \frac{2\pi}{t} [x(\varphi) - x(\varphi')] - \dot{x}(\varphi) \sin \frac{2\pi}{t} [y(\varphi) - y(\varphi')]}{\cosh \frac{2\pi}{t} [x(\varphi) - x(\varphi')] - \cos \frac{2\pi}{t} [y(\varphi) - y(\varphi')]} \frac{\pi}{t} + \frac{\pi}{t} \dot{y}(\varphi) \quad (3)$$

The limit value of this function when  $\varphi' \rightarrow \varphi$  is

$$\mathcal{K}(\varphi, \varphi) = \lim_{\varphi' \rightarrow \varphi} \mathcal{K}(\varphi, \varphi') = \frac{1}{2} \frac{\ddot{y}(\varphi) \dot{x}(\varphi) - \ddot{x}(\varphi) \dot{y}(\varphi)}{\dot{x}^2(\varphi) + \dot{y}^2(\varphi)} + \frac{\pi}{t} \dot{y}(\varphi) \quad (4)$$

The right-hand side of Equ. (2) can be composed of three parts

$$F(\varphi) = F_1(\varphi) \tan \chi_1 + F_2(\varphi) + \omega [F_{30}(\varphi) + F_{31}(\varphi)] \quad (5)$$

where

$$F_1(\varphi) = 2\dot{y}(\varphi) ; \quad F_2(\varphi) = 2\dot{x}(\varphi) ;$$

$$F_{30}(\varphi) = \frac{2\pi}{Nt} \left[ \frac{2r_1^2 - r^2(\varphi)}{w_{1x}} + \frac{NA_{pox}}{\int w_{1x}} \right] \dot{y}(\varphi) ; \quad F_{31}(\varphi) = \frac{2\pi}{Nt^2} \int_0^{2\pi} \frac{r^2(\varphi')}{w_{1x}} G(\varphi, \varphi') d\varphi' ; \quad (6)$$

$$G(\varphi, \varphi') = -G(\varphi', \varphi) = \frac{[\dot{x}(\varphi)\dot{x}(\varphi') - \dot{y}(\varphi)\dot{y}(\varphi')] \sinh \frac{2\pi}{t} [x(\varphi) - x(\varphi')] + [\dot{y}(\varphi)\dot{x}(\varphi') + \dot{x}(\varphi)\dot{y}(\varphi')] \sin \frac{2\pi}{t} [y(\varphi) - y(\varphi')]}{\cosh \frac{2\pi}{t} [x(\varphi) - x(\varphi')] - \cos \frac{2\pi}{t} [y(\varphi) - y(\varphi')]}$$

Since  $G(\varphi, \varphi')$  has a first order singularity at  $\varphi' = \varphi$ , the Cauchy's principal value of the integral in  $F_{31}(\varphi)$  must be taken. Note that if the cascade is either stationary or axial the last term in Equ. (5) vanishes. In the first case  $\omega = 0$  and in the second one the sum in brackets becomes zero. In both cases the computational work is being reduced to that of a straight cascade.

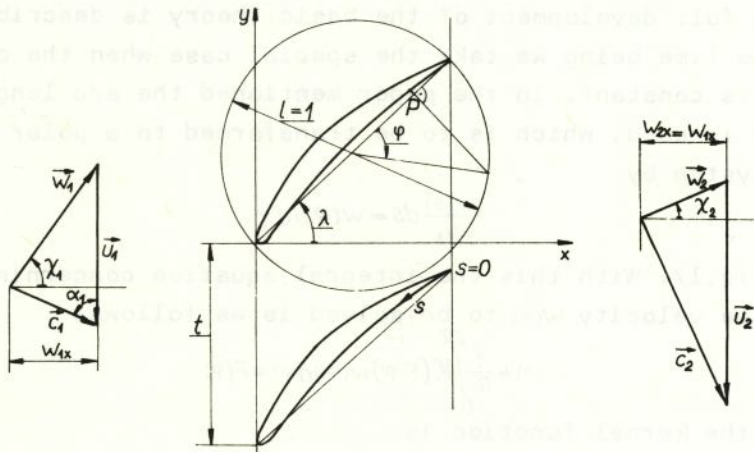


Fig.1

Introducing the trapezoidal integration with  $N1$  pivotal points /even number/ Equ. (2) becomes

$$\sum_{j=1}^{N1} K_{ij} w_j = F_{1i} \tan \chi_1 + F_{2i} + \omega (F_{30i} + F_{31i}) \quad (i = 1, 2, \dots, N1) \quad (7)$$

where

$$K_{ij} = \frac{1}{\pi} \int_{(\Delta\varphi_j)} \mathcal{K}(\varphi_i, \varphi) d\varphi + \delta_{ij} \quad (i, j = 1, 2, \dots, N1) ; \quad (8)$$

$$w_j = w(\varphi_j) ; \quad F_i = F(\varphi_i) ; \quad \varphi_j = (j - 0,5) \frac{2\pi}{N1} ; \quad (9)$$

$\Delta\varphi_j = 2\pi/N1$  and  $w(\varphi)$  is considered to be constant over a  $(\Delta\varphi_j)$  interval. The choice of such a distribution of the pivotal points results in a proper density in the vicinity of the edges /Fig.2/. The kernel function (3) is very steep when the sum of  $i$  and  $j$  is near to  $N1$  i.e. the points are on the opposite side of the profile. The problem arising due to the steepness of  $\mathcal{K}(\varphi, \varphi')$  is overcome by applying Equ.(8) otherwise

$$K_{ij} = \frac{2}{N1} \mathcal{K}(\varphi_i, \varphi_j) + \delta_{ij} \quad (10)$$

by which sufficient accuracy is attained. Using Equ.(10) only would lead to a poor solution particularly in the case of thin foils.

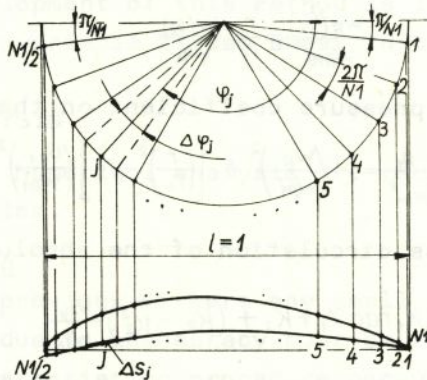


Fig.2

Taking into consideration the results in paper [6] the sum of each column of matrix  $K_{ij}$  is zero. Thus the system of linear algebraic equations (7) is indeterminate. Applying the Kutta-Joukowski condition in the form below

$$w_{N1} = -\alpha w_1 \quad (11)$$

- where  $\alpha \cong 1$  - the system (7) becomes determinate:

$$\sum_{j=1}^{N-1} K_{ij}' w_j = F_i \quad (i=1, 2, \dots, N-1) \quad (12)$$

where

$$K_{ij}' = K_{ij} - K_{iN} \delta_{jN} \quad (i, j=1, 2, \dots, N-1) \quad (13)$$

The meaning of condition (11) is that the stagnation point is near to the trailing edge. We note that the matrix  $K'_{ij}$  is a well conditioned one.

The system of equations (12) is to be solved with three different RHS /see Equ. (7)/. The transformed velocity can be written at the incidence  $\chi_1$  and angular velocity  $\omega$  as follows:

$$w_i = w_{1i} \tan \chi_1 + w_{2i} + \omega w_{3i} \quad (14)$$

where  $w_{1i}$ ,  $w_{2i}$ ,  $w_{3i}$  are the three different solutions of Equ. (12). Using transformation (1) the contour velocity as a function of arc length is

$$\frac{w(s_i)}{w_{1x}} = \frac{w_i}{\sqrt{x_i^2 + y_i^2}} \quad (i=1, 2, \dots, N) \quad (15)$$

and on the surface of revolution is:

$$\frac{w_R(s_i)}{w_{R16}} = \frac{r_i}{r_1} \frac{w(s_i)}{w_{1x}} \quad (16)$$

furthermore the pressure coefficient on the same surface is:

$$C_p = \frac{p - p_1}{\frac{\rho}{2} w_{R1}^2} = 1 - \left( \frac{w_R}{w_{R1}} \right)^2 + \left[ \left( \frac{r}{r_1} \right)^2 - 1 \right] \left( \frac{w_{R1}}{w_{R1}} \right)^2 \quad (17)$$

The dimensionless circulation of the absolute velocity is

$$\frac{\Gamma}{t w_{1x}} = K_1 \tan \chi_1 + K_2 + (K_3 - \kappa_2^2) \frac{U_{2y}}{w_{1x}} \quad (18)$$

where

$$K_{1,2} = \frac{2\pi}{t N} \sum_{i=1}^{N-1} w_{1,2i} \quad \text{and} \quad K_3 = \frac{N w_{1x}}{r_2^2 N} \sum_{i=1}^{N-1} w_{3i} \quad (19)$$

In case of an axial cascade

$$K_3 = 0 \quad \text{and} \quad \kappa_2^2 = \frac{A_{pax} N}{r_2^2 \pi} = 0 \quad (20)$$

while  $u_{2y} = 0$  stands for a stationary cascade.

By using the well known relationship between the circulation and the pressure number  $\Psi$  and flow number  $\Phi$  the theoretical

characteristics of a pump is:

$$\psi_p = 2(K_2 - K_1 \cot \alpha_1) \phi_p + 2 \left[ K_1 \left( \frac{r_1}{r_2} \right)^2 + u_2^2 - K_3 \right] \quad (21)$$

and those of a turbine

$$\psi_T = 2(K_2 + K_1 \cot \alpha_1) \phi_T + 2 \left[ -K_1 + (K_3 - u_2^2) \left( \frac{r_2}{r_1} \right)^2 \right] \quad (22)$$

The outlet angle  $\chi_2$  /Fig.1/ for a pump is given by the relation

$$\tan \chi_2 = \left[ 1 - K_1 \left( \frac{r_1}{r_2} \right)^2 + K_3 - u_2^2 \right] \frac{1}{\phi_p} + (K_1 - 1) \cot \alpha_1 - K_2 \quad (23)$$

and for a turbine

$$\tan \chi_2 = \left[ K_1 - (1 + K_3 - u_2^2) \left( \frac{r_2}{r_1} \right)^2 \right] \frac{1}{\phi_T} + (1 - K_1) \cot \alpha_1 - K_2 \quad (24)$$

This method of solution and the computer program can also be used to determine the flow around a single foil if we put  $t/l \rightarrow \infty$  /in practice  $t/l \geq 10^4$ /.

Further development of this method is in progress for the cascade flow with variable stream sheet thickness.

## 2. NUMERICAL ANALYSIS

Six areas of numerical analysis are required in the solution of this problem.

### a/ Data smoothing

As shown by previous workers any small perturbations in the profile data due to inaccuracy must be removed before entering the differentiating procedure because differentiation amplifies errors, leading to poor solution. The powerful method in [8] is used in which method the graduated values obtained minimize both their  $r$ -th differences and their deviations from the values to be smoothed simultaneously. In the program  $r = 3$ .

### b/ Numerical differentiation

To obtain derivatives for values given at equidistant points the five point formula is used:

$$60hf_k' = f_{k+3} - f_{k-3} - 9(f_{k+2} - f_{k-2}) + 45(f_{k+1} - f_{k-1})$$

c/ Evaluating the Cauchy's principal value for the integral

$$I = \int_{x - \frac{\Delta x}{2}}^{x + \frac{\Delta x}{2}} \frac{f(x')}{x - x'} dx' = I_0 + I_1 \quad (25)$$

where

$$I_0 = \int_{x - \frac{\Delta x}{2}}^{x - \varepsilon} \frac{f(x')}{x - x'} dx' + \int_{x + \varepsilon}^{x + \frac{\Delta x}{2}} \frac{f(x')}{x - x'} dx' ,$$

$$I_1 = -2 \sum_{n=0}^{\infty} f^{(2n+1)}(x) \frac{\varepsilon^{2n+1}}{(2n+1)(2n+1)!}$$

The latter expression is obtained by expanding  $f(x')$  into Taylor series around  $x$  and performing the integral in (25) over the interval  $[x - \varepsilon, x + \varepsilon]$  ( $\varepsilon < \Delta x/2$ ; small positive number). The integration sets up from the endpoints of interval  $[x - \Delta x/2, x + \Delta x/2]$  and tends towards the pole until the condition

$$|2\varepsilon f'(x)| < E$$

is satisfied, where  $E$  is the error bound. As a result we have

$$I \cong I_0 - 2\varepsilon f'(x)$$

d/ Interpolation

- Newton's general interpolation formula /divided differences/
- Hermite interpolation

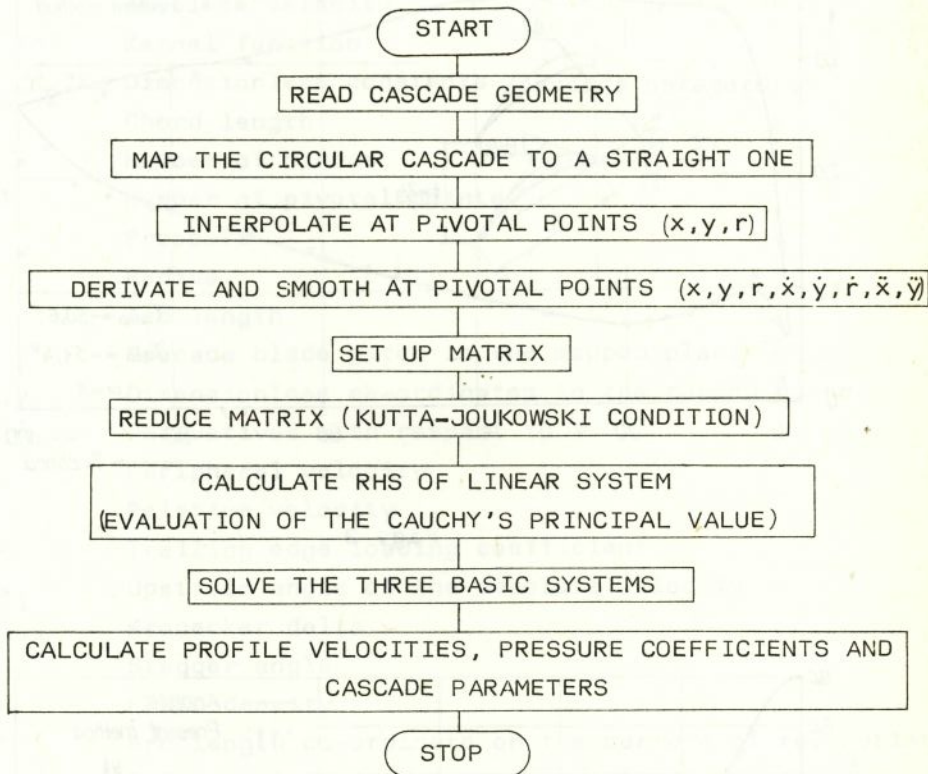
e/ Numerical integration

- The rectangle rule
- Simpson's rule
- Romberg's method

f/ Solving the set of linear equations

A subroutine which provides iterative improvement of solution is used available at the Dep. of Computing Techniques at the Technical University in Miskolc.

### 3. FLOW CHART OF COMPUTER PROGRAM



### 4. SAMPLE CALCULATIONS

The computer program has been tested against a series of axial turbine cascades where test results are available [3], and a number of analytic solutions relating to single foils. The agreement between the theoretical and experimental /or analytic/ distributions is quite satisfactory. Due to lack of space only two examples are shown /see Fig.3, Fig.4/.

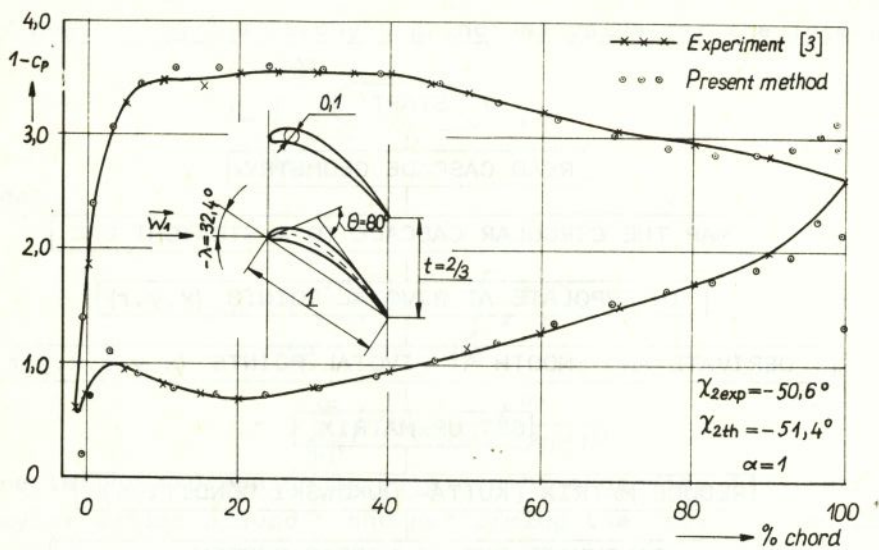


Fig. 3

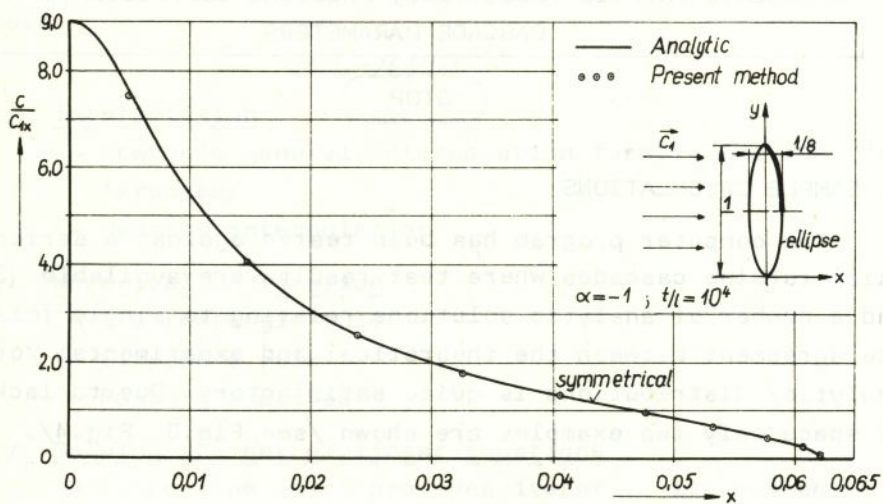


Fig. 4

## NOTATION

$A_{pax}$	Area of a foil's axial projection
$c_p$	Pressure coefficient
$c$	Absolute velocity
$K(\varphi, \varphi')$	Kernel function
$K_1, K_2, K_3$	Dimensionless constants /cascade parameters/
$l$	Chord length
$N$	Number of blades
$Nl$	Number of pivotal points
$p$	Pressure
$r$	Radius
$s$	Arc length
$t$	Cascade blade pitch in the mapped plane
$x, y$	Dimensionless co-ordinates in the mapped plane
$\dot{x}, \dot{y}, \dots$	Derivatives with respect to $\varphi$
$u$	Peripheral velocity
$w$	Relative velocity
$\alpha$	Trailing edge loading coefficient
$\alpha_1$	Upstream angle of the absolute velocity
$\delta_{ij}$	Kronecker delta
$\lambda$	Stagger angle
$\rho$	Fluid density
$\sigma$	Arc length co-ordinate on the surface of revolution
$\varphi$	Polar angle
$\phi$	Flow number
$\chi$	Angle included by a direction with the x-axis
$\psi$	Pressure number
$\omega$	Angular velocity

### Subscripts

$i, j$	Locations in a profile contour
$P$	Pump
$R$	On the surface of revolution
$T$	Turbine
$x, y$	Components in the corresponding co-ordinate directions
$\infty$	Infinity
$1, 2$	Upstream and downstream, respectively

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