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Determination of the potential plane flow around bodies by means of two solutions belonging to different incidence angles

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DETERMINATION OF THE POTENTIAL PLANE FLOW AROUND BODIES BY MEANS OF TWO SOLUTIONS BELONGING TO DIFFERENT INCIDENCE ANGLES

by
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In this paper an integral equation is derived for the potential plane flow past a fixed cylindrical body of arbitrary cross-section, placed in a uniform flow. It follows from the features of the integral equation that its solution is completely determined by that of the ones belonging to two approaching flows. As an application of this the derivation of the lift coefficient in a simple form is also presented.

Main Symbols

(B)	Boundary curve of the cross-section of the body
c_L	Lift coefficient
$\overline{c}$	Conjugate complex velocity, $c_x - ic_y$
L	Representative length
$\mathcal{K}(z', z)$	Kernel function defined by equations (15) and (13)
x, y	Planar Cartesian co-ordinates
z	Complex variable, $x + iy$

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α	Angle of incidence of uniform stream $\mathbf{c}_\infty$
Γ	Circulation
χ	Angle included with the x-direction

Subscripts

B	On boundary (B)
s	Tangential to curve (B)
n	Normal to curve (B)
x, y	Components in the corresponding co-ordinate directions
∞	Infinity

Let us consider the potential plane flow past a fixed rigid body immersed in a uniform flow of fluid not subject to external body forces. The intensity of the frictionless, incompressible flow field is $\mathbf{c}_\infty$. It is assumed that the flow is completely attached to the body, that is, it does not separate from the surface.

Before studying the problem in detail let us show an important feature of the conjugate complex velocity

$$\overline{c}(z) = c_x - ic_y \,. \qquad (1)$$

For this purpose the *Cauchy–Riemann* equations are written for the real and imaginary parts of $\overline{c}(z)$

$$\frac{\partial c_x}{\partial x} = -\frac{\partial c_y}{\partial y} \qquad (2)$$

$$\frac{\partial c_x}{\partial y} = \frac{\partial c_y}{\partial x} \,. \qquad (3)$$

Relation (2) is the equation of continuity for incompressible two-dimensional flow, and equation (3) expresses the condition for the vortex free plane flow. Since both conditions (2) and (3) prevail in the points of the flow space, the conjugate complex velocity $\overline{c}(z)$ is an analitic function of the complex variable $z = x + iy$.

Therefore, the *Chauchy*'s theorem can be applied to $\overline{c}(z)$ in the single connected domain, bounded by the curve (b) (*Fig. 1*). This gives

$$\overline{c}(z) = \frac{1}{2\pi i} \oint_{(b)} \frac{\overline{c}(z')dz'}{z' - z} \,. \qquad (4)$$

In *Fig. 1* (R) is a circle of radius $R \to \infty$ and the centre z. Along the circle (R) $\overline{c}(z) = \overline{c}_\infty$, furthermore the integral taken along the splitting curve (m) vanishes. Hence, after resolving integral (4) into three, we obtain

$$\overline{c}(z) = \overline{c}_\infty + \frac{1}{2\pi i} \oint_{(B)} \frac{\overline{c}(z')dz'}{z' - z} \,. \qquad (5)$$

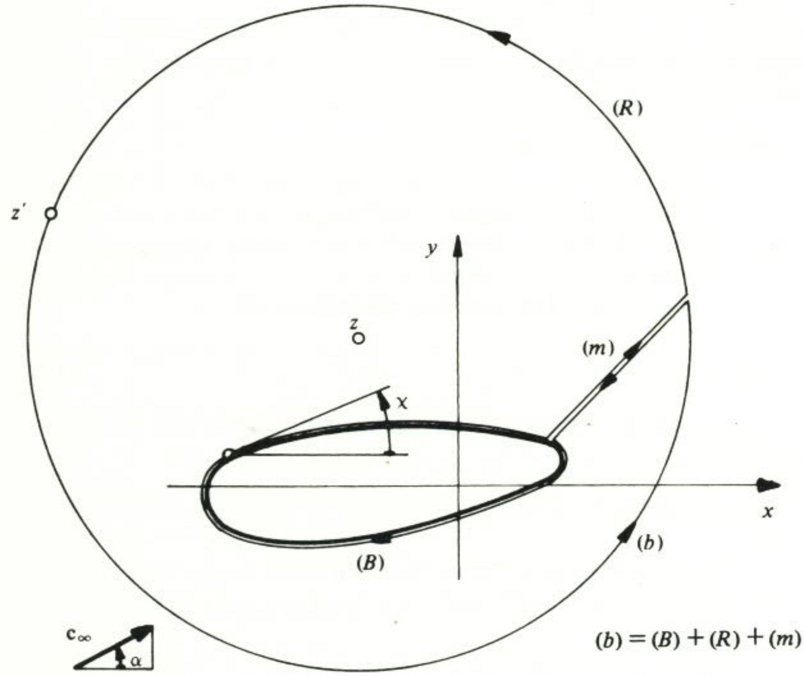

Fig. 1. Region of the flow

In the knowledge of the velocity distribution along boundary (B), this formula can be used to compute the velocity in the points not belonging to contour (B).

Whenever the variable point of integration z', however, tends to the contour point z_B, by virtue of the *Plemelj*'s formula we can write

$$\bar{c}^{+}(z_B) = \bar{c}_\infty + \frac{1}{2}\bar{c}(z_B) + \frac{1}{2\pi i}\oint_{(B)} \frac{\bar{c}(z')dz'}{z' - z_B}\,, \tag{6}$$

$$\bar{c}^{-}(z_B) = \bar{c}_\infty - \frac{1}{2}\bar{c}(z_B) + \frac{1}{2\pi i}\oint_{(B)} \frac{\bar{c}(z')dz'}{z' - z_B}\,. \tag{7}$$

In these equations the superscripts + and − refer to cases when $z' \to z_B$ while z' lies within or outside curve (b), respectively. By virtue of the *Cauchy*'s integral formula $\bar{c}^{+}(z_B) = \bar{c}(z_B)$ and $\bar{c}^{-}(z_B) = 0$. If equations (6) and (7) are combined the following integral equation is obtained

$$\bar{c}(z_B) = 2\bar{c}_\infty + \frac{1}{\pi i} \oint_{(B)} \frac{\bar{c}(z')dz'}{z' - z_B} \ . \tag{8}$$

The conjugate complex velocity $\bar{c}(z_B)$ can be resolved into tangential and normal components on boundary (B)

$$\bar{c}(z_B) = (c_s - ic_n)e^{-i\chi} \tag{9}$$

where χ is the angle between the tangent of contour (B) and the x-axis (*Fig. 1*). Since the body surface is a streamsurface, the normal velocity component c_n vanishes identically on it. From equation (8) the integral equation for the unknown tangential velocity component c_s may be obtained in two ways. The first possibility

$$\text{Im}\,[\bar{c}(z_B)\,e^{i\chi}] = -c_n = 0\ . \tag{10}$$

In this case the integral equation sought for c_s, has a singular kernel. The other possibility is furnished by the relation

$$\text{Re}\,[\bar{c}(z_B)e^{i\chi}] = c_s\ . \tag{11}$$

This latter one leads to a *Fredholm* integral equation of the second kind [1], [2]. In equations (10) and (11) Im[ ] and Re[ ] denote the imaginary and real parts of the function in the brackets, respectively.

To reshape our equations let us introduce the notation

$$K(z', z) = \frac{1}{z' - z} \tag{12}$$

for the function occuring in equations (5) and (8). Furthermore, let us resolve the kernel function into a real and an imaginary part

$$K(z', z) = R(z', z) + iJ(z', z) \tag{12a}$$

where

$$R(z', z) = \frac{x' - x}{(x' - x)^2 + (y' - y)^2}$$

and (13)

$$J(z', z) = -\frac{y' - y}{(x' - x)^2 + (y' - y)^2}\ .$$

By multiplying the equation (8) by $e^{i\chi}$ and further using that

$$dz' = ds'\,e^{i\chi}\ ,$$

according to equation (11) the following integral equation may be obtained

$$0 = c_{\infty x} \cos\chi + c_{\infty y} \sin\chi - \frac{1}{2} c_s(z_B) +$$

$$+ \frac{1}{2\pi} \oint_{(B)} c_s(z') \left[R(z', z_B) \sin\chi + J(z', z_B) \cos\chi \right] ds' \,, \tag{14}$$

where ds is the element of arc length along the boundary (B). In fact equation (14) simply states that the velocity just inside the inner region at z_B is zero.

By employing the notation

$$\mathcal{K}(z', z_B) = R(z', z_B) \sin\chi + J(z', z_B) \cos\chi \tag{15}$$

equation (14) may be written as follows

$$-\frac{1}{2} \frac{c_s(z_B)}{c_{\infty x}} + \frac{1}{2\pi} \oint_{(B)} \frac{c_s(z')}{c_{\infty x}} \mathcal{K}(z', z_B)\, ds' = -\cos\chi - \sin\chi \tan\alpha \,. \tag{16}$$

The kernel function $\mathcal{K}(z', z_B)$ is everywhere bounded. Let us calculate now the limit value of this function as $z' \to z_B$ [1]:

$$\lim_{z' \to z_B} \mathcal{K}(z', z_B) = \sin\chi \lim_{z' \to z_B} \frac{x' - x_B - (y' - y_B)\cot\chi}{(x' - x_B)^2 + (y' - y_B)^2} =$$

$$= \sin\chi \lim_{z' \to z_B} \frac{1 - \dfrac{y' - y_B}{x' - x_B} \cot\chi}{\left[1 + \left(\dfrac{y' - y_B}{x' - x_B} \right)^2 \right] (x' - x_B)} \,.$$

Taking into account that along the contour (B):

$$\frac{y' - y_B}{x' - x_B} = \tan\chi + \frac{1}{2} \frac{d^2 y_B}{dx_B^2} (x' - x_B) + o\left[(x' - x_B)^2 \right]$$

therefore

$$\frac{y' - y_B}{x' - x_B} \cot\chi = 1 + \frac{1}{2} \frac{d^2 y_B}{dx_B^2} (x' - x_B) \cot\chi + o\left[(x' - x_B)^2 \right] .$$

Making use of this we have

$$\lim_{z' \to z_B} \mathcal{K}(z', z_B) = -\frac{1}{2} \cos^3\chi \frac{d^2 y_B}{dx_B^2} = -\frac{1}{2r_c} \tag{17}$$

where r_c is the radius of the curvature.

Now let us see an important feature of the solution of equation (16). NYÍRI [1] has proved that the flow pattern of an arbitrary approach around a turbomachine cascade is the linear combination of the two solutions belonging to two different approaches. The solution of the potential plane flow around an infinite cylindrical body of arbitrary cross-section has the same property. For the sake of brevity let us introduce the notation $t = \tan\alpha$ and let there be $c_s^*/c_{\infty x}^*$ the solution of the integral equation (16) for $t^* = \tan\alpha^*$, and the solution $c_s^0/c_{\infty x}^0$ shall belong to the incidence $t^0 = \tan\alpha^0$.

Statement: the function

$$\frac{c_s(z_B)}{c_{\infty x}} = \frac{t-t^0}{t^*-t^0}\,\frac{c_s^*(z_B)}{c_{\infty x}^*} + \frac{t^*-t}{t^*-t^0}\,\frac{c_s^0(z_B)}{c_{\infty x}^0} \tag{18}$$

describes the velocity field around the body belonging to the incidence $t = \tan\alpha$.

To prove this statement let us write the integral equation (16) formally for $t^* = \tan\alpha^*$ and for $t^0 = \tan\alpha^0$. By multiplying the first of these by $(t-t^0)/(t^*-t^0)$ and the second one by $(t^*-t)/(t^*-t^0)$ then summing them up we obtain

$$-\frac{1}{2}\left[\frac{t-t^0}{t^*-t^0}\,\frac{c_s^*(z_B)}{c_{\infty x}^*} + \frac{t^*-t}{t^*-t^0}\,\frac{c_s^0(z_B)}{c_{\infty x}^0}\right] +$$

$$+\frac{1}{2\pi}\oint_{(B)}\left[\frac{t-t^0}{t^*-t^0}\,\frac{c_s^*(z')}{c_{\infty x}^*} + \frac{t^*-t}{t^*-t^0}\,\frac{c_s^0(z')}{c_{\infty x}^0}\right]\mathcal{K}(z', z_B)\,ds' =$$

$$= -\cos\chi\left[\frac{t-t^0}{t^*-t^0} + \frac{t^*-t}{t^*-t^0}\right] - \sin\chi\left[t^*\frac{t-t^0}{t^*-t^0} + t^0\,\frac{t^*-t}{t^*-t^0}\right]. \tag{19}$$

Since the expression in the first brackets on the left-hand side of equation (19) equals $c_s(z_B)/c_{\infty x}$, in the integrand $c_s(z')/c_{\infty x}$, the first brackets on the right-hand side of (19) equals unity, and the value of the term in the second brackets is precisely t. Therefore expression (18) is a solution of the integral equation (16), indeed, as it was stated.

This feature of the solution means that from the solutions of the potential plane flow past an infinite cylindrical body for two different incidence, the solution for an arbitrary incidence can be calculated by their linear combination. Hence, the dimensionless velocity $c_s/c_{\infty x}$ at a contour point is a linear function of the tangent of incidence α (*Fig. 2*).

By using this property of the solution, the line integral of the velocity distribution, viz. the circulation Γ can be written as

$$\frac{\Gamma}{c_{\infty x}} = \oint_{(B)} \frac{c_s}{c_{\infty x}}\,ds = a\tan\alpha + b \tag{20}$$

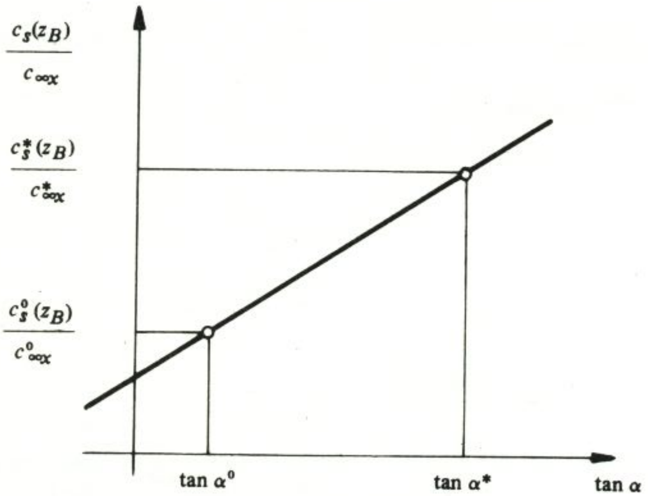

Fig. 2. Tangential velocity versus incidence

where

$$a = \frac{\dfrac{\Gamma^*}{c^*_{\infty x}} - \dfrac{\Gamma^0}{c^0_{\infty x}}}{t^* - t^0} \; ; \qquad b = \frac{t^* \dfrac{\Gamma^0}{c^0_{\infty x}} - t^0 \dfrac{\Gamma^*}{c^*_{\infty x}}}{t^* - t^0} \, . \tag{21}$$

It is obvious from equation (20) that the circulation divided by $c_{\infty x}$ is also a linear function of $\tan \alpha$.

By virtue of the *Joukowski*'s theorem for the lift force per unit length F_L can be expressed similarly

$$\frac{F_L}{c_{\infty x}} = \frac{\rho \Gamma c_\infty}{c_{\infty x}} = \rho \, c_\infty \, (a \tan \alpha + b) \tag{22}$$

where the constants a and b are defined by equation (21), and ρ is the density of the fluid. The lift coefficient c_L has more practical interest than the lift force F_L has, and is defined by

$$c_L = \frac{F_L}{L \dfrac{\rho}{2} c_\infty^2} \tag{23}$$

where L is a representative length of the body's cross-section. By inserting equation (22) into (23) we obtain

$$c_L = \frac{2}{L}\,(a \sin\alpha + b \cos\alpha) \tag{24}$$

which can be written in the form:

$$c_L = k \sin(\alpha + \beta) \tag{25}$$

where

$$k = \frac{2\sqrt{a^2 + b^2}}{L}$$

$$\beta = \text{arc}\sin\left(\frac{b}{\sqrt{a^2 + b^2}}\right). \tag{26}$$

Numerical solutions of equation (16) may be found by introduction of the trapezoidal integration based upon a selection of pivotal points on the body contour (see e. g. [4] where this method of integration is used for the computation of the flow past annular aerofoils and bodies of revolution). In the knowledge of the solution of equation (16), equation (5) may be used to compute the velocity at any point outside contour (B). We do not wish to deal with these problems here.

It has been assumed that the flow will not separate from the body surface. Hence, this theory can only mean a good approximation of the flow pattern for a streamlined body and for a limited range of incidence.

To illustrate the above mentioned theory a well-known, simple example will be shown: the flow past circular cylinder (*Fig. 3*). Here R is the radius of the cylinder, furthermore the angles φ and φ' relate to the point under consideration $P(x_B, y_B)$ and the variable point of integration $P'(x', y')$ respectively. It is clear from *Fig. 3* that

$$x_B = R\cos\varphi \;, \qquad x' = R\cos\varphi'$$

$$y_B = -R\sin\varphi\,, \qquad y' = -R\sin\varphi'$$

$$\chi = \frac{3}{2}\pi - \varphi\,,$$

$$ds' = R\,d\varphi'\,.$$

Using these relations, the functions $R(z', z_B)$ and $J(z', z_B)$ (13) can be written in the form

$$R(z', z_B) = \frac{\cos\varphi' - \cos\varphi}{2R[1 - \cos(\varphi' - \varphi)]}$$

$$J(z', z_B) = \frac{\sin\varphi' - \sin\varphi}{2R[1 - \cos(\varphi' - \varphi)]} \quad .$$

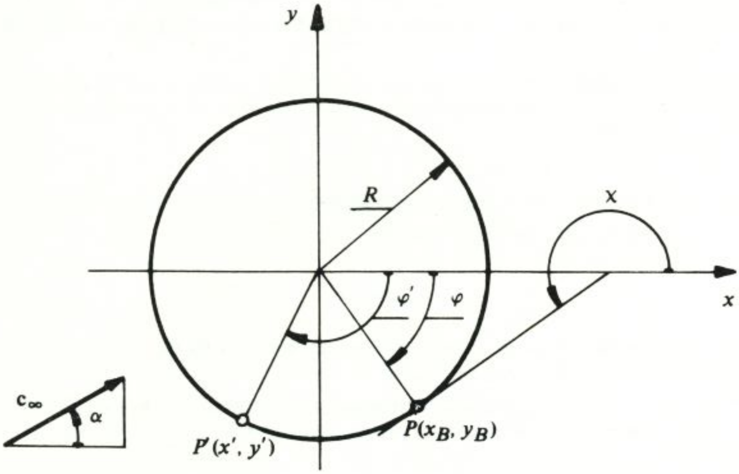

Fig. 3. To the flow past a circular cylinder

Then, after a simple calculation we find that the kernel function in equation (15) is a constant

$$\mathcal{K}(z', z_B) = \frac{1}{2R} \quad .$$

Hence, the integral equation reduces to

$$-\frac{1}{2}\ \frac{c_s(\varphi)}{c_{\infty x}} + \frac{1}{4\pi} \int_{\varphi'=0}^{2\pi} \frac{c_s(\varphi')}{c_{\infty x}}\, d\varphi' = \sin\varphi + \cos\varphi \tan\alpha\,. \tag{27}$$

The solution of this equation may be expected in the form

$$\frac{c_s(\varphi)}{c_{\infty \chi}} = -2\,(\sin\varphi + \cos\varphi \tan\alpha) + \text{const.} \tag{28}$$

This function satisfies equation (16) for arbitrary value of the const. The real solution can be choosen from these infinite set of solutions by imposing the condition that the tangential velocity c_s equals zero at $\varphi_1 = \pi - \alpha$, and it is

$$\frac{c_s(\varphi)}{c_{\infty x}} = -2(\sin\varphi + \cos\varphi \tan\alpha)\,. \tag{29}$$

It is obvious from this solution that the dimensionless velocity $c_s/c_{\infty x}$ is a linear function of the tangent of the incidence angle α. After multiplying equation (29) by $\cos\alpha$ and introducing a new variable ψ instead of φ with the relation $\psi = -\varphi$ the well-known form of the solution is obtained

$$\frac{c_s(\psi)}{c_\infty} = 2\sin(\psi - \alpha)\,. \tag{30}$$

It is obvious from this solution that the line integral of the velocity along the circle i. e. the circulation vanishes. Hence, the lift force F_L is zero, too.

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BESTIMMUNG DER EBENEN POTENTIALUMSTRÖMUNG EINES KÖRPERS MIT HILFE VON ZU ZWEI UNTERSCHIEDLICHEN STRÖMUNGSRICHTUNGEN GEHÖRENDEN LÖSUNGEN

von

L. BARANYI

Zusammenfassung

Eine Integralgleichung wurde abgeleitet für die ebene Potentialströmung, die um den in Parallelströmung gesetzten unendlich langer zylindrischer Körper mit arbiträren Querschnitt entsteht. Aus den Eigenschaften dieser folgt, daß die zu zwei unterschiedlichen Strömungsrichtungen gehörenden Lösungen die Umströmung des Körpers völlig bestimmen. Als Anwendung demonstrierten wir die einfache Herstellung des Auftriebskoeffizienten.

ОПРЕДЕЛЕНИЕ ПЛОСКОГО, БЕЗВИХРЕВОГО ДВИЖЕНИЯ ВОКРУГ ТЕЛА С ПОМОЩЬЮ РЕШЕНИЯ, ОТНОСЯЩЕГОСЯ К ДВУМ РАЗЛИЧНЫМ УГЛАМ АТАКИ

Л. БАРАНИ

Резюме

В работе выводен интегральное уравнение плоского безвихревого движения вокруг цилиндрического тела, бесконечной длины и любого сечения расположенного в плоскопараллельном потоке. Из качеств интегрального уравнения следует, что решение, относящееся к двум различным углам атаки, полностью определяет обтекание тела. В качестве применения представляется простое получение коэффициента подъемной силы.